AD-A118 446 VIRGINIA POLYTECHNIC INST AND STATE UNIV BLACKSBURG --ETC F/G 1/1
HEREDITARY MODELS FOR AIRFOILS IN UNSTEADY AERODYNAMICS, NUMERI--ETC(U)
FEB 82 J A BURNS, E M CLIFF AFOSR-80-0068
UNCLASSIFIED VPI-E-81-18 AFWAL-TR-81-3173 NL

1 OF 1
AD
A118446

END
DATE
FILMED
9-82
DTIC

AFWAL-TR-81-3173

HEREDITARY MODELS FOR AIRFOILS IN UNSTEADY AERODYNAMICS, NUMERICAL APPROXIMATIONS AND PARAMETER ESTIMATION

J. A. Burns, Department of Mathematics
E. M. Cliff, Aerospace and Ocean Engineering

Virginia Polytechnic Institute and State University
Blacksburg, Virginia 24061-4097

FEBRUARY 1982

FINAL REPORT FOR PERIOD 15 DECEMBER 1979 - 15 MARCH 1981

Approved for public release; distribution unlimited

FLIGHT DYNAMICS LABORATORY
AIR FORCE WRIGHT AERONAUTICAL LABORATORIES
AIR FORCE SYSTEMS COMMAND
WRIGHT-PATTERSON AFB, OHIO 45433

NOTICE

When Government drawings, specifications, or other data are used for any purpose other than in connection with a definitely related Government procurement operation, the United States Government thereby incurs no responsibility nor any obligation whatsoever; and the fact that the government may have formulated, furnished, or in any way supplied the said drawings, specifications, or other data, is not to be regarded by implication or otherwise as in any manner licensing the holder or any other person or corporation, or conveying any rights or permission to manufacture use, or sell any patented invention that may in any way be related thereto.

This report has been reviewed by the Office of Public Affairs (ASD/PA) and is releasable to the National Technical Information Service (NTIS). At NTIS, it will be available to the general public, including foreign nations.

This technical report has been reviewed and is approved for publication.

Robert C Schwanz

ROBERT C. SCHWANZ, Ph.D.
Project Engineer
Control Dynamics Branch
Flight Control Division

R O Anderson

R. O. ANDERSON, Chief
Control Dynamics Branch
Flight Control Division

FOR THE COMMANDER

Ernest F. Moore

ERNEST F. MOORE, Colonel, USAF
Chief, Flight Control Division

"If your address has changed, if you wish to be removed from our mailing list, or if the addressee is no longer employed by your organization please notify AFWAL/FIGC, *W-PAFB, OH 45433 to help us maintain a current mailing list".*

Copies of this report should not be returned unless return is required by security considerations, contractual obligations, or notice on a specific document.

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER AFWAL-TR-81-3173	2. GOVT ACCESSION NO. AD-A118 446	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) HEREDITARY MODELS FOR AIRFOILS IN UNSTEADY AERODYNAMICS, NUMERICAL APPROXIMATIONS AND PARAMETER ESTIMATION		5. TYPE OF REPORT & PERIOD COVERED Final report covering period from 12/15/79 through 3/15/81. 6. PERFORMING ORG. REPORT NUMBER VPI-E-81-18
7. AUTHOR(s) J. A. Burns E. M. Cliff		8. CONTRACT OR GRANT NUMBER(s) AFOSR-80-0068
9. PERFORMING ORGANIZATION NAME AND ADDRESS Virginia Polytechnic Institute & State University Blacksburg, Virginia 24061 - 4097		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS PE 61102F 2307 N3 22
11. CONTROLLING OFFICE NAME AND ADDRESS Flight Dynamics Laboratory (AFWAL/FIGC) AF Wright Aeronautical Laboratories, AFSC Wright-Patterson AFB, Ohio 45433		12. REPORT DATE February 1982 13. NUMBER OF PAGES 81
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office)		15. SECURITY CLASS. (of this report) Unclassified 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Hereditary Models
Unsteady Aerodynamics
Parameter Estimation
Numerical Approximations

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

This report is concerned with the development of a general mathematical model for the motion of a two-dimensional airfoil in unsteady aerodynamics and with the study of numerical schemes for estimating parameters in this model. The equations of motion are a set of coupled ordinary and functional differential equations. Approximating Wagner functions are used to construct state space models that retain the hereditary nature of unsteady fluid flow. The results of some simple numerical experiments are presented including a preliminary analysis of wind-tunnel data.

DD FORM 1 JAN 73 1473 EDITION OF 1 NOV 65 IS OBSOLETE

Unclassified
SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

FORWARD

This report describes work performed at Virginia Polytechnic Institute and State University for the Flight Control Division, Control Dynamics Branch, Flight Dynamics Laboratory, Air Force Wright Aeronautical Laboratories, Wright-Patterson Air Force Base, Ohio. The purpose of the work was to formulate dynamic models suitable for use in parameter identification studies of a simple aero-elastic system. Dr. Robert C. Schwanz, Control and Analysis Group, was the Technical Monitor. The work was performed under Grant AFOSR-80-0068, project element 61102F, project 2307, Mechanics Task 2307N3, Dynamics of Aerospace Vehicles, work unit 2307N322. The Principal Investigators were Dr. John A. Burns, Department of Mathematics, and Dr. Eugene M. Cliff, Aerospace and Ocean Engineering Department. The authors gratefully achknowledge the help of Dr. S. C. McIntosh, Nielson Research and Engineering Company, Mountain View, California and Dr. W. P. Rodden, Consultant, La Canãda, California. Dr. McIntosh furnished some wind-tunnel data analyzed in Section 6, while Dr. Rodden provided some assistance in the model formulation.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF ILLUSTRATIONS

SECTION 1

INTRODUCTION

In this report we present a general model for the motions of a two-dimensional airfoil in unsteady aerodynamics. The equations of motion are a set of coupled ordinary and functional differential equations. Making use of general "approximating Wagner functions" we develop a state space model that can be used for identification and optimal control of such systems.

In Section 2 we formulate the basic (hereditary) equations and in Section 3 we develop the state space model. Although the resulting model is infinite dimensional, there are certain special features of the state space model that can be exploited in order to construct approximation schemes. In Section 4 we discuss these features and present an approximation technique that can be used for parameter estimation and optimal control of such systems. Section 5 contains numerical examples that illustrates the basic ideas. Section 6 presents some preliminary numerical results from an analysis of data derived from wind-tunnel tests on an oscillating airfoil. Throughout this report we shall use the symbol

$$\int_a^b d_s\, f(s)\, g(s)$$

to denote the Riemann-Stieltjes integral of g with respect to f over the interval [a,b] (see [6] for a discussion of Riemann-Stieltjes integration).

SECTION 2

THE HEREDITARY MODEL

Consider the two-dimensional airfoil shown in Figure 1.1. The basic equations for the pitching and plunging motions of the airfoil may be found in a number of papers (see for example [10,17]). We shall follow the development in [17] and use fairly standard nomenclature and conventions. We shall not write down the exact constants which may also be found in the various references cited previously.

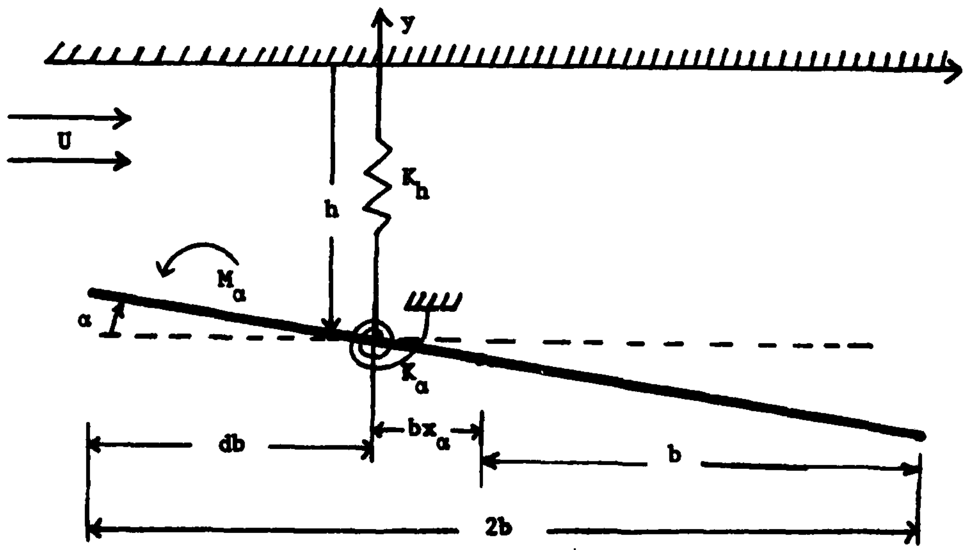

FIGURE 1.1. The typical airfoil.

Let $h(t)$ denote the "plunge" and $\alpha(t)$ the "pitch" of the airfoil at time t. The equations of motion can be written in the form

$$M_s \ddot{z}(t) + K_s z(t) = -F(t), \tag{1}$$

where $z(t)$ is the column vector $z(t) = \mathrm{col}[h(t) \quad \alpha(t)]$ and F contains the aerodynamic loads on the airfoil. In particular,

$$F(t) = \begin{bmatrix} L(t) \\ M_\alpha(t) \end{bmatrix}, \tag{2}$$

where L and M_α are the aerodynamic loads corresponding to total wing lift per unit depth and total moment about the 1/4 chord per unit depth, respectively. For the airfoil considered here, it follows that (note that in this case $d = x_\alpha = 1/2$)

$$M_\alpha(t) = \pi\rho b^3 \{[\tfrac{1}{2} \quad \tfrac{3b}{8}]\ \ddot{z}(t) + [0 \quad U]\ \dot{z}(t)\}, \tag{3}$$

and

$$L(t) = \pi\rho b^2 \{[1 \quad \tfrac{b}{2}]\ \ddot{z}(t) + [0 \quad U]\ \dot{z}(t)\} \tag{4}$$

$$+ (2\pi\rho U b)\ D(t),$$

where D(t) is the "Duhamel integral." In particular, D(t) is given by

$$D(t) = \int_0^t \phi(\tfrac{U}{b}(t - \tau))\ \dot{Q}(\tau)d\tau, \tag{5}$$

where $\Phi(Ut/b)$ is the *Wagner* function, and

$$Q(t) = \int_0^t \{[1\ b]\ \ddot{z}(\tau) + [0\ U]\ \dot{z}(\tau)\}d\tau. \tag{6}$$

To obtain a state space model that is suitable for identification and optimal control, one must provide a "useful" representation for D(t). One approach to this problem is to approximate D(t) by approximating the Wagner function Φ. For example, the Jones two-term exponential approximation of the Wagner function (see [13,14]) results in a finite dimensional model where D(t) may be viewed as the output of a second order linear control system. Recently, it has been shown that D(t) may be realized as the output of an infinite dimensional control system (see [1, 2, 10]). In particular, D(t) can be explicitly represented as the output of a system governed by a functional differential equation (i.e., a hereditary system, see [9]). If this explicit representation is exploited, then the resulting model becomes an infinite dimensional control system. Again, it will be necessary to introduce some type of approximation in order to obtain a workable finite dimensional model.

Although we shall make use of the fact that D(t) is the output of an hereditary system, the approach presented below is similar in *spirit* to the Jones two-term exponential approximation of the Wagner function which we now briefly describe.

Jones [13] approximated the Wagner function Φ by the two-term exponential function

$$\tilde{\Phi}(\frac{Ut}{b}) = 1 - \alpha_1 e^{-\beta_1 Ut/b} - \alpha_2 e^{-\beta_2 Ut/b}, \tag{7}$$

where $\alpha_1 = 0.165$, $\alpha_2 = 0.335$, $\beta_1 = 0.0455$ and $\beta_2 = 0.3$. If $\tilde{\Phi}$ is substituted into (5), then D(t) is approximated by

$$\tilde{D}(t) = Q(t) - \alpha_1 B_1(t) - \alpha_2 B_2(t), \tag{8}$$

where B_1, B_2 satisfy the ordinary differential equations

$$\begin{cases} \dot{B}_1(t) = -\beta_1 \frac{U}{b} B_1(t) + \dot{Q}(t) \\ \dot{B}_2(t) = -\beta_2 \frac{U}{b} B_2(t) + \dot{Q}(t) \end{cases} . \tag{9}$$

This approach leads to a sixth order ordinary differential equation model for the motion of the airfoil (see [17] for complete details).

In order to understand the hereditary model to be discussed below, it is worthwhile to look at the same approximation scheme (i.e., the approximating model resulting from equations (7)-(9)) from a different (system theory) point of view. Write the Wagner function as

$$\Phi(\frac{U}{b}t) = 1 - W(t), \tag{10}$$

and observe that the Duhamel integral (5) is equal to

$$D(t) = Q(t) - y(t) \tag{11}$$

where

$$y(t) = \int_0^t W(t - \tau)\ \dot{Q}(\tau)d\tau \ . \tag{12}$$

The function W(t) is called a *weighting pattern* and the Jones two-term exponential approximation (7) is equivalent to approximating W(t) by

$$\tilde{W}(t) = [\alpha_1\ \alpha_2]\ e^{U/b \begin{bmatrix} -\beta_1 & 0 \\ 0 & -\beta_2 \end{bmatrix} t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} . \tag{13}$$

Observe that (13) can be written as

$$\tilde{W}(t) = C\ \tilde{X}(t)\ B\ ,$$

where X(t) is the fundamental matrix for the ordinary differential equation (of dimension 2)

$$\dot{\tilde{x}}(t) = \frac{U}{b} \begin{bmatrix} -\beta_2 & 0 \\ 0 & -\beta_2 \end{bmatrix} \tilde{x}(t); \tag{14}$$

$C = [\alpha_1\ \alpha_2]$ and $B = \mathrm{col}[1\ 1]$. In particular,

$$\tilde{D}(t) = Q(t) - \tilde{y}(t) \tag{15}$$

where $\tilde{y}(t)$ is the output of the second order control system

$$\dot{\tilde{x}}(t) = \frac{U}{b} \begin{bmatrix} -\beta_1 & 0 \\ 0 & -\beta_2 \end{bmatrix} \tilde{x}(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix} \dot{Q}(t) \tag{16}$$

with output

$$\tilde{y}(t) = [\alpha_1 \ \alpha_2] \ \tilde{x}(t) \tag{17}$$

Consequently, one may view the Jones approximation as an approximating of

$$y(t) = \int_0^t W(t - \tau) \ \dot{Q}(\tau) d\tau$$

by the output of the (second order) finite dimensional system (16)-(17) with input $\dot{Q}(t)$.

In the model detailed below, we propose to approximate $y(t)$ by the output of a simple hereditary system (i.e., a retarded functional differential equation). There are two basic reasons for considering this approach. As indicated above, one can show that the model should include hereditary terms (see [2], [9]). Moreover, since ordinary differential systems may be viewed as "special" functional differential equations, the inclusion of delayed terms is a very natural extension of the "exponential type" approximations.

We turn now to the general problem of approximating $y(t)$ (and hence $D(t)$) by the output of a simple hereditary system. For each parameter $p \in \mathbb{R}^{\mu}$, let $G(t,p)$ denote a $n \times n$ matrix valued function such that for $p \in \mathbb{R}^{\mu}$ the mapping $t \to G(t,p)$ is of bounded variation on compact subintervals of $(-\infty,0]$. Let B and C be n-dimensional column and row vectors, respectively. Let $r > 0$ and consider the n-dimensional retarded functional differential equation

$$\dot{x}_d(t) = \int_{-r}^{0} d_s G(s,p) x_d(t+s) + B\dot{Q}(t). \tag{18}$$

The output of system (18) is defined by

$$y_d(t) = C\, x_d(t). \tag{19}$$

The system defined by (18)-(19) is parameterized by $\gamma = (p,r,B,C^T)$. We define Ω to be the set of all parameters $\gamma = (p,r,B,C^T) \in \mathbb{R}^{\mu} \times \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$ such that $r > 0$ and the pair B, C satisfies

$$CB = \sum_{i=1}^{n} c_i b_i = 1/2 . \tag{20}$$

The set Ω is called the set of *admissible parameters* for (18)-(19).

Given (p,r), let $X(t;p,r)$ denote the $n \times n$ matrix satisfying

$$\begin{cases} \dot{X}(t) = \int_{-r}^{0} d_s G(s,p) X(t+s) ds, & t \geq 0, \\ X(0) = I_n, & \\ X(s) = 0_n, & s < 0, \end{cases} \tag{21}$$

where I_n and 0_n denote the $n \times n$ identity and zero matrices, respectively. The matrix $X(t;p,r)$ is called the *fundamental matrix* for (18). (see [7, 12]). Given $\gamma = (p,r,B,C^T) \in \Omega$, define the *approximate Wagner function* $\Phi(Ut/b:\gamma)$ by

$$\phi(\frac{Ut}{b};\gamma) = 1 - W(t;\gamma), \tag{22}$$

where $W(t;\gamma)$ is the weighting pattern for the hereditary system (18)-(19) given by

$$W(t;\gamma) = C\,X(t;p,r)\,B\,. \tag{23}$$

Observe that the condition that $CB = 1/2$ implies that $\phi(U0/b;\gamma) = 1/2$. Moreover, if $y(t;\gamma)$ denotes the output of (18)-(19), then it follows that (see Hale [12])

$$y(t;\gamma) = \int_0^t W(t-\tau;\gamma)\,\dot{Q}(\tau)d\tau. \tag{24}$$

Let $D(t;\gamma)$ and $L(t;\gamma)$ be defined by (see equations (4)-(5))

$$D(t;\gamma) = \int_0^t \phi(\frac{U}{b}(t-\tau))\,\dot{Q}(\tau)d\tau = Q(t) - y(t;\gamma) \tag{25}$$

and

$$L(t;\gamma) = \pi\rho b^2\,\{[1\ \ \frac{b}{2}]\,\ddot{z}(t) + [0\ \ U]\,\dot{z}(t)\} \tag{26}$$

$$+\ (2\pi\rho Ub)\,D(t;\gamma),$$

respectively. If $L(t;\gamma)$ is substituted for $L(t)$ in (1)-(2), then we obtain the model

$$M_s\,\ddot{z}(t) + K_s\,z(t) = -F(t;\gamma) \tag{27}$$

where

$$F(t;\gamma) = \begin{bmatrix} L(t;\gamma) \\ M_\alpha(t) \end{bmatrix}. \qquad (28)$$

SECTION 3

THE STATE SPACE MODEL

Given the parameter $\gamma = (p, r, B, C^T) \in \Omega$ we shall construct a dynamical system for the airfoil described in Figure 1.1. Recall that $z(t) = \mathrm{col}[h(t) \quad \alpha(t)]$ and $x_d(t)$ is the n-dimensional vector function that satisfies the retarded functional differential equation (18). Let $x(t)$ denote the $(4 + n)$-dimensional vector

$$x(t) = \mathrm{col}[\ \dot{h}(t) \ \ \dot{\alpha}(t) \ \ h(t) \ \ \alpha(t) \ \ x_d(t)]. \tag{29}$$

The basic model is defined by ((27)-(28))

$$M_s\, \ddot{z}(t) + K_s\, z(t) = -F(t;\gamma) \tag{30}$$

where

$$F(t,\gamma) = \begin{bmatrix} L(t;\gamma) \\ M_\alpha(t) \end{bmatrix} = \pi\rho b^2 \left\{ \begin{bmatrix} 1 & \frac{b}{2} \\ \frac{b}{2} & \frac{3b^2}{8} \end{bmatrix} \ddot{z}(t) + \begin{bmatrix} 0 & U \\ 0 & Ub \end{bmatrix} \dot{z}(t) \right\} \tag{31}$$

$$+ (2\pi\rho Ub) \begin{bmatrix} 1 \\ 0 \end{bmatrix} D(t;\gamma) .$$

Observe that

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix} D(t;\gamma) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \{Q(t) - y(t;\gamma)\}$$

and under the assumption that $\dot{z}(0) = z(0) = 0$, it follows that

$$\begin{bmatrix}1\\0\end{bmatrix} D(t;\gamma) = \begin{bmatrix}1 & b\\0 & 0\end{bmatrix}\dot{z}(t) + \begin{bmatrix}0 & U\\0 & 0\end{bmatrix} z(t) - \begin{bmatrix}1\\0\end{bmatrix} y(t;\gamma). \quad (32)$$

Using the fact that $y(t;\gamma) = C\,x_d(t)$, it follows from equations (31) and (32) that (30) is equivalent to

$$M_s\,\ddot{z}(t) + K_s\,z(t) = -\pi\rho b^2 \begin{bmatrix}1 & \frac{b}{2}\\ \frac{b}{2} & \frac{3b^2}{8}\end{bmatrix}\ddot{z}(t) - \pi\rho b^2 \begin{bmatrix}0 & U\\0 & Ub\end{bmatrix}\dot{z}(t) \quad (33)$$

$$- 2\pi\rho Ub \begin{bmatrix}1 & b\\0 & 0\end{bmatrix}\dot{z}(t) - 2\pi\rho Ub \begin{bmatrix}0 & U\\0 & 0\end{bmatrix} z(t)$$

$$+ 2\pi\rho Ub \begin{bmatrix}C\\0\end{bmatrix} x_d(t) .$$

Define the 2×2 matrices E_{11}, A_{11} and A_{12} by

$$E_{11} = \left\{ M_s + \pi\rho b^2 \begin{bmatrix}1 & \frac{b}{2}\\ \frac{b}{2} & \frac{3b^2}{8}\end{bmatrix} \right\}, \quad (34)$$

$$A_{11} = -\left\{ \pi\rho b^2 \begin{bmatrix}0 & U\\0 & Ub\end{bmatrix} + 2\pi\rho Ub \begin{bmatrix}1 & b\\0 & 0\end{bmatrix} \right\}, \quad (35)$$

and

$$A_{12} = -\left\{ K_s + 2\pi\rho Ub \begin{bmatrix}0 & U\\0 & 0\end{bmatrix} \right\}, \quad (36)$$

respectively. If A_{13} denotes the $2 \times n$ matrix

$$A_{13} = 2\pi\rho Ub \begin{bmatrix} C \\ 0 \end{bmatrix} = 2\pi\rho Ub \begin{bmatrix} C_1 & C_2 & \cdots & C_n \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \tag{37}$$

then the system (33) can be written as

$$E_{11}\, \ddot{z}(t) = A_{11}\, \dot{z}(t) + A_{12}\, z(t) + A_{13}\, x_d(t). \tag{38}$$

Moreover, $x_d(t)$ satisfies

$$\dot{x}_d(t) = \int_{-r}^{0} d_s G(s,p) x_d(t+s) + B\,\dot{Q}(t) \tag{39}$$

$$= \int_{-r}^{0} d_s G(s,p) x_d(t+s) + B\{[1\ b]\ddot{z}(t) + [0\ U]\dot{z}(t)\}.$$

Define the $n \times 2$ matrices E_{31} and A_{31} by

$$E_{31} = -B[1\ b] = -\begin{bmatrix} b_1 & bb_1 \\ b_2 & bb_2 \\ \vdots & \vdots \\ b_n & bb_n \end{bmatrix}, \tag{40}$$

and

$$A_{31} = B[0\ U] = \begin{bmatrix} 0 & Ub_1 \\ 0 & Ub_2 \\ \vdots & \vdots \\ 0 & Ub_n \end{bmatrix}, \tag{41}$$

respectively. With E_{31} and A_{31} so defined, equation (39) becomes

$$E_{31}\, \ddot{z}(t) + \dot{x}_d(t) = \int_{-r}^{0} d_s G(s,p) x_d(t+s) + A_{31}\, \dot{z}(t). \tag{42}$$

In terms of the "state vector" $x(t) = \mathrm{col}[\dot{z}(t) \;\; z(t) \;\; x_d(t)]$, equations (38) and (42) combine to form the system

$$E\dot{x}(t) = Ax(t) + \int_{-r}^{0} d_s H(s,p)\, x(t+s), \qquad (43)$$

where E, A and H(s,p) are defined by

$$E = \begin{bmatrix} E_{11} & 0 & 0 \\ 0 & I_2 & 0 \\ E_{31} & 0 & I_n \end{bmatrix}, \qquad (44)$$

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ I_2 & 0 & 0 \\ A_{31} & 0 & 0 \end{bmatrix}, \qquad (45)$$

and

$$H(s,p) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & G(s,p) \end{bmatrix}. \qquad (46)$$

Observe that in general (43) is a linear retarded functional differential equation with finite delay $r > 0$. However, the model that results from the Jones two-term exponential approximation is but a special case of (43).

EXAMPLE 3.1. Let $n = 2$, $p = (-\beta_1, -\beta_2)$ and $C = [\alpha_1 \;\; \alpha_2]$. Fix $r > 1$ and let $B = \mathrm{col}[1 \;\; 1]$. Define the 2×2 matrix-valued function $\bar{G}(s,p)$ by

$$\tilde{G}(s,p) = \begin{cases} \frac{U}{b}\begin{bmatrix} -\beta_1 & 0 \\ 0 & -\beta_2 \end{bmatrix}, & \text{if } s = 0, \\ \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, & \text{if } s < 0 . \end{cases}$$

If $x_d(t)$ is any continuous 2-dimensional vector function defined on $[-r, +\infty)$, then

$$\int_{-r}^{0} d_s\tilde{G}(s,p)x_d(t + s) = \frac{U}{b}\begin{bmatrix} -\beta_1 & 0 \\ 0 & -\beta_2 \end{bmatrix}x_d(t),$$

and the "delay equation" (39) reduces to the ordinary differential equation

$$\dot{x}_d(t) = \frac{U}{b}\begin{bmatrix} -\beta_1 & 0 \\ 0 & -\beta_2 \end{bmatrix}\dot{x}_d(t) + \begin{bmatrix} 1 \\ 1 \end{bmatrix}\dot{Q}(t).$$

The output in this case is given by

$$y(t;\gamma) = [\alpha_1 \quad \alpha_2]x_d(t),$$

and the corresponding fundamental matrix $\tilde{X}(t) = X(t;\gamma)$ is given by

$$X(t;\gamma) = \begin{bmatrix} e^{-U/b\ \beta_1 t} & 0 \\ 0 & e^{-U/b\ \beta_2 t} \end{bmatrix}.$$

Consequently, the approximating Wagner function defined by (20)-(23) is given by

$$\tilde{\Phi}(\frac{U}{b}t) = \Phi(\frac{U}{b}t;\gamma) = 1 - [\alpha_1 \quad \alpha_2]\begin{bmatrix} e^{-U/b\ \beta_1 t} & 0 \\ 0 & e^{-U/b\ \beta_1 t}\end{bmatrix}\begin{bmatrix}1\\1\end{bmatrix}$$

$$= 1 - \alpha_1 e^{-U/b\ \beta_1 t} - \alpha_2 e^{-U/b\ \beta_2 t},$$

which is the Jones two-term exponential approximation. Therefore, the general model (43) includes as special cases those models derived from two-term "Jones type" approximations.

EXAMPLE 3.2. Assume that the airfoil represents a tail section of a flexible aircraft and that this tail section is in the wake of the forward wing. The aerodynamic loads on the tail at time t are influenced by the motion of the forward wing at some previous time, say $(t-r)$. Moreover, because of the elastic coupling between the tail and the wing, the motion of the forward wing is influenced by the loads on the tail. Consequently, the loads on the tail at time t involve the motions of the tail at a previous time $(t-r)$. This "time delay" feature of a wing-tail configuration has been observed by Reding and Ericsson [14,15] in their stability analysis of the 747/Orbiter.

To see how one can incorporate these pure time delays in the model defined by (3.15) we assume that $r > 0$ and $n \geq 1$ are fixed. Let B and C be such that $CB = 1/2$ and for $p \in \mathbb{R}^{\mu}$ assume that $A_0(p)$, $A_1(p)$ are $n \times n$ matrices parameterized by p. Define the $n \times n$ matrix function $G(s,p)$ by

$$G(s,p) = \begin{cases} A_0(p) + A_1(p), & s = 0, \\ A_1(p), & -r < s < 0, \\ 0, & s \le -r. \end{cases} \qquad (47)$$

If $x_d(t)$ is a continuous n-dimensional vector function, then

$$\int_{-r}^{0} d_s G(s,p) x_d(t + s) = A_0(p)x_d(t) + A_1(p)x_d(t - r).$$

Consequently, the functional differential equation (18) becomes the delay-differential equation

$$\dot{x}_d(t) = A_0(p)x_d(t) + A_1(p)x_d(t - r) + B\dot{Q}(t).$$

Also, the basic model (43) is equivalent

$$E\dot{x}(t) = F_0(p)x(t) + F_1(p)\, x(t - r)$$

where E is defined by (44) and $F_0(p)$ and $F_1(p)$ are given by

$$F_0(p) = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ I_2 & 0 & 0 \\ A_{31} & 0 & A_0(p) \end{bmatrix}, \qquad (48)$$

and

$$F_1(p) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A_1(p) \end{bmatrix}, \qquad (49)$$

respectively.

We conclude this section with a model that includes a pure time delay as well as distributed delays. This model includes all of the previously discussed models and yet a number of nice features are retained which make the model a useful tool for identification and optimal control.

Let $r > 0$ and $n \geq 1$ be fixed and CB = 1/2 be as above. Assume that $G_c(s,p)$ is an $n \times n$ matrix function continuous in s and p and define $G(s,p)$ by

$$G(s,p) = G_a(s,p) + \int_{-r}^{s} G_c(u,p)du,$$

where $G_a(s,p)$ is defined by (47), i.e.

$$G_a(s,p) = \begin{cases} A_0(p) + A_1(p), & s = 0, \\ A_1(p), & -r < s < 0, \\ 0, & s \leq -r. \end{cases} \tag{50}$$

If $G(s,p)$ is defined as above, and $x_d(t)$ is a continuous n-dimensional vector function, then

$$\int_{-r}^{0} d_s G(s,p)x_d(t+s) = A_0(p)x_d(t)$$

$$+ A_1(p)x_d(t-r) + \int_{-r}^{0} G_c(s,p)x_d(t+s)ds.$$

Consequently, equation (43) may be written as

$$E\dot{x}(t) = F_0(p)x(t) + F_1(p)x(t-r) + \int_{-r}^{0} K(s,p)x(t+s)ds, \tag{51}$$

where E is defined by (44), $F_0(p)$, $F_1(p)$ are defined by (48)-(49) and $K(s,p)$ is given by

$$K(s,p) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & G_c(s,p) \end{bmatrix}. \tag{52}$$

Although the model described by (51) has the nice feature that it in some sense generalizes the classical approach due to Jones [13], there are also theoretical justifications for considering (51) as the basic model. In particular, the "delayed state" $x_d(t)$ is closely related to the circulation function $\Gamma(t)$ (see [4,9]) and it may be shown using theoretical aerodynamics that $\Gamma(t)$ does in fact satisfy a linear functional differential equation. A complete discussion of these theoretical aspects will appear in a forthcoming paper (see [9]).

The structure of the system (51) is very special. There are four "non-delayed states" ($\dot{h}(t)$, $\dot{\alpha}(t)$, $h(t)$, $\alpha(t)$) and n "delayed states" $x_d(t)$. More precisely, the matrices $F_1(p)$ and $K(s,p)$ consist of zeros except in the lower right-hand $n \times n$ blocks. This particular feature can be exploited in the development of numerical algorithms, resulting in enormous savings in storage and CPU time. In the next section we present two numerical schemes for identification and control of systems governed by equations of the form (51). These schemes are modified versions of the so called AVE and SPLINE schemes (see [3,4,5]) that have been developed especially for general hereditary systems of the form (51).

SECTION 4

THE REDUCED STATE MODEL AND NUMERICAL APPROXIMATIONS

In this section we consider the hereditary control system governed by the linear functional differential of order (4 + n)

$$E(p)\dot{x}(t) = F_0(p)x(t) + F_1(p)x(t - r) \tag{53}$$

$$+ \int_{-r}^{0} K(s,p)x(t + s)ds + Gu(t),$$

with initial data

$$x(0) = \eta, \qquad x(s) = \phi(s), \qquad s < 0, \tag{54}$$

and output

$$y(t) = H\,x(t), \tag{55}$$

where η is a (4 + n)-dimensional vector and $\phi(t)$ is a (4 + n)-dimensional vector valued function. Moreover, we assume that the system matrices $E(p)$, $F_1(p)$ and $K(s,p)$ have the special forms,

$$E(p) = \begin{bmatrix} E_{11}(p) & 0 & 0 \\ 0 & I_2 & 0 \\ E_{31}(p) & 0 & I_n \end{bmatrix}, \tag{56}$$

$$F_1(p) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & A_1(p) \end{bmatrix}, \tag{57}$$

and

$$K(s,p) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & G_c(s,p) \end{bmatrix} . \tag{58}$$

We assume that for each $p \in \mathbb{R}^\mu$ the $n \times n$ matrix valued function $s \to G_c(s,p)$ is integrable on $[-r,0]$ and the 2×2 matrix $E_{11}(p)$ is non-singular.

Since $E_{11}(p)$ is non-singular, it follows that $E(p)$ is non-singular and

$$[E(p)]^{-1} = \begin{bmatrix} [E_{11}(p)]^{-1} & 0 & 0 \\ 0 & I_2 & 0 \\ -E_{31}(p)[E_{11}(p)]^{-1} & 0 & I_n \end{bmatrix} . \tag{59}$$

The basic idea is to approximate the hereditary system (53)-(55) by a (possibly large) system of ordinary differential equations. This approach to identification and optimal control of hereditary systems has been used by a number of investigators (see [3] for a survey of this technique).
There are two particular schemes that have been used for such hereditary systems; the *AVE* scheme (a "finite difference" type scheme to be described below) and the *SPLINE* scheme (a "finite element" type scheme, see [4,5]).

Although these two schemes have been applied to a number of hereditary systems, there is a problem with the procedure. In particular, the size of the approximating system of ordinary differential equations can become very large and thus reducing the effectiveness of the algorithms. However, the special form of the hereditary system (53)-(55) can be exploited to greatly improve the usefulness of these general algorithms. To describe the necessary modifications, we shall concentrate on the *AVE* scheme. Similar modifications are valid for the *SPLINE* scheme and a complete theoretical and numerical analysis of both procedures (including convergence results) will appear in a future paper.

Let $0 < r_m < r_M < +\infty$ and suppose that $\Gamma \subseteq \mathbb{R}^{\mu} \times [r_m, r_M]$ is a compact convex set. We assume that the initial data $\eta \in \mathbb{R}^{(4+n)}$ and $\phi: [-r_n, 0] \to \mathbb{R}^{(4+n)}$ are fixed and consider the following typical parameter estimation problem (see [8]).

PROBLEM (ID). *Given the input function* u *and observations* $\bar{y}_i$ *at times* $\bar{t}_i$ $(0 \leq \bar{t}_1 < \bar{t}_2 < \ldots < \bar{t}_m \leq T)$, *find the system parameters* $\gamma^* = (p^*, r^*) \in \Gamma$ *which minimize the error*

$$J(\gamma) = \frac{1}{2} \sum_{i=1}^{n} \| y(\bar{t}_i;\gamma) - \bar{y}_i \|^2, \tag{60}$$

where $y(t;\gamma)$ *is the output at time* t *to the hereditary system* (53)-(55).

Given $\gamma = (p,r) \in \Gamma$ and a positive integer N, partition the interval $[-r,0]$ into N subintervals $[t_j^N(r), t_{j-1}^N(r)]$, where $t_j^N(r) = -jr/N$, $j = 0, 1, 2, \ldots, N$. For $j = 0, 1, 2, \ldots, N$ define ϕ_j^N by

$$\phi_0^N = \eta, \tag{61}$$

and

$$\phi_j^N = \frac{N}{r} \int_{t_j^N(r)}^{t_{j-1}^N(r)} \phi(s)ds. \tag{62}$$

Note that ϕ_j^N is an $(4 + n)$-dimensional vector and that

$$x_j^N = \text{col}\,[\phi_0^N, \phi_1^N, \ldots, \phi_N^N] \tag{63}$$

is an $(N + 1)(4 + n)$-dimensional vector.

Define the $[(N + 1)(4 + n)] \times [(N + 1)(4 + n)]$ matrices $E^N(p)$ and $A^N(p)$ by

$$E^N(p) = \begin{bmatrix} E(p) & 0 \\ 0 & 0 \end{bmatrix}, \tag{64}$$

and

$$
A^N(p) = \begin{bmatrix}
F_0(p) & K_1^N(p) & K_2^N(p) & \cdots & K_{N-1}^N(p) & F_1(p) + K_N^N(p) \\
\frac{N}{r} I & \frac{-N}{r} I & 0 & \cdots & 0 & 0 \\
0 & \frac{N}{r} I & \frac{-N}{r} I & \cdots & 0 & 0 \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdots & \cdot & \cdot \\
0 & 0 & \cdot & \cdots & \frac{N}{r} I & \frac{-N}{r} I
\end{bmatrix} \tag{65}
$$

where

$$
K_j^N(p) = \int_{t_j^N(r)}^{t_{j-1}^N(r)} K(s,p)ds. \tag{66}
$$

Let H^N and G^N be the $(N + 1)(4 + n)$-dimensional row and column vectors defined by

$$
H^N = [H, 0, 0, \ldots, 0] \tag{67}
$$

and

$$
G^N = \text{col } [G, 0, 0, \ldots, 0], \tag{68}
$$

respectively.

With $E^N(p)$, $A^N(p)$, x_0^N, H^N and G^N defined by (61)-(68) above, the N^{th} *approximating control system* is defined by

$$
E^N(p)\dot{x}^N(t) = A^N(p)x^N(t) + G^N u(t) \tag{69}
$$

with initial condition

$$x^N(0) = x_0^N \tag{70}$$

and output

$$y^N(t) = H^N x^N(t). \tag{71}$$

Corresponding to system (69)-(71) we consider the following approximating parameter estimating problem.

PROBLEM (IDN). *Given the input function* u *and observations* $\bar{y}_i$ *at times* $\bar{t}_i$ $(0 \le \bar{t}_1 < \bar{t}_2 < \ldots < \bar{t}_m \le T)$, *find the system parameters* $\hat{\gamma}^N = (\hat{p}^N, \hat{r}^N) \in \Gamma$ *which minimize the error*

$$J^N(\gamma) = \frac{1}{2} \sum_{i=1}^{M} \| y^N(\bar{t}_i;\gamma) - \bar{y}_i \|^2, \tag{72}$$

where $y^N(t;\gamma)$ *is the output at time* t *to the* N^{th} *approximating system* (69)-(71).

Under very reasonable assumptions it is possible to show that as $N \to \infty$, the optimal solutions $\hat{\gamma}^N$ to PROBLEM (IDN) converge to γ^*, the optimal solution of PROBLEM (ID) (see [3,8]). Therefore, the idea is to select a value of N, construct the approximating system (69)-(71) and use a standard algorithm to solve PROBLEM (IDN) for $\hat{\gamma}^N$. If N is "large enough", then $\hat{\gamma}^N$ is a good approximation to γ^*. This particular numerical scheme is called the AVE scheme.

Consider the size of the approximating system (69)-(71) defined by the AVE scheme above. The dimension of this system is $DIM(N,n) = (N + 1)(4 + n)$, which for large values of N (or n) grows very rapidly. This is illustrated in Table 4.1 for various values of N and for n = 1, 2, 3.

TABLE 4.1. The dimension of the AVE approximating systems for values of N and n.

N	DIM(N,1)	DIM(N,2)	DIM(N,3)
2	15	18	21
4	25	30	35
8	45	56	63
16	85	102	119
32	165	198	231
64	325	390	455

For N = 16, 32, 64 . . . , the size of the approximating system (69)-(71) becomes an important (and limiting) consideration in numerical computations. Therefore, we seek some method to reduce this computational burdon. To modify the AVE scheme it is sufficient to make a simple but important observation concerning the special structure of the hereditary equation (53). In particular, we note that at most n of the (4 + n) states in (53) are delayed. Consequently it is not necessary to approximate all of the "history" $x(t - r)$, but only those coordinates which are delayed.

Let n_d denote the maximu[illegible]umber of non-zero columns in $A_1(p)$ and the number of non-zero columns in $G_c(s,p)$. The number n_d must fall between 0 and n. If $n_d = 0$, then $A_1(p) = 0$ and $G_c(s,p) \equiv 0$ so that (53) is an ordinary differential equation and no approximation is needed. Consequently, we assume that $1 \leq n_d \leq n$ and partition the matrices $F_0(p)$, $F_1(p)$ and $K(s,p)$ as follows:

$$F_0(p) = [F_{01}(p) \vdots F_{02}(p)], \tag{73}$$

$$F_1(p) = [0 \vdots F_{12}(p)], \tag{74}$$

$$K(s,p) = [0 \vdots K_2(s,p)], \tag{75}$$

where $F_{02}(p)$, $F_{12}(p)$ and $K_2(s,p)$ are $[(4 + n) \times n_d]$ matrices. Here we have assumed without loss of generality that the first $(4 + n)-n_d$ columns are the zero columns. Let $[0 \vdots I_{n_d}]$ denote the $n_d \times (4 + n)$ matrix consisting of the matrix with zeros in the first $(4 + n)-n_d$ columns and define ψ_j^N, $j = 0, 1, 2, \ldots, N$ by

$$\psi_0^N = \eta \tag{76}$$

and

$$\psi_j^N = [0 \vdots I_{n_d}]\phi_j^N \tag{77}$$

where ϕ_j^N is defined by (62). Note that

$$\Psi_0^N = \text{col}\,[\psi_0^N, \psi_1^N, \ldots, \psi_N^N] \tag{78}$$

is an $[n_d \cdot N + (4 + n)]$-dimensional vector.

Define the $[n_d \cdot N + (4 + n)] \times [n_d \cdot N + (4 + n)]$ matrices $E_R^N(p)$ and $A_R^N(p)$ by

$$E_R^N(p) = \begin{bmatrix} E(p) & 0 & 0 & \ldots & 0 \\ 0 & I_{n_d} & 0 & \ldots & 0 \\ \cdot & \cdot & I_{n_d} & \ldots & 0 \\ \cdot & \cdot & & \ddots & \cdot \\ \cdot & \cdot & & & \cdot \\ 0 & 0 & \ldots & & I_{n_d} \end{bmatrix} \tag{79}$$

and

$$A_R^N(p) = \begin{bmatrix} F_{01}(p) & F_{02}(p) & K_{21}^N(p) & K_{22}^N(p) & \ldots & K_{2N-1}^N(p) & F_{12}(p) + K_{2N}^N \\ 0 & \frac{N}{r} I_{n_d} & -\frac{N}{r} I_{n_d} & 0 & \ldots & 0 & 0 \\ 0 & 0 & \frac{N}{r} I_{n_d} & -\frac{N}{r} I_{n_d} & \ldots & 0 & 0 \\ \cdot & \cdot & & \ddots & & & \cdot \\ \cdot & \cdot & & & \ddots & & \cdot \\ \cdot & \cdot & & & & \ddots & \cdot \\ \cdot & \cdot & & & & & \cdot \\ 0 & 0 & \cdot & \cdot & \cdot & \frac{N}{r} I_{n_d} & -\frac{N}{r} I_{n_d} \end{bmatrix} \tag{80}$$

where I_{n_d} represents the $n_d \times n_d$ identity and

$$K_{2j} = \int_{t_j^N(r)}^{t_{j-1}^N(r)} K_2(s,p)ds. \tag{81}$$

The fact that only n_d of the states in the hereditary equation (53) are delayed implies that the state space model defined by (53)-(55) is not a "minimal realization" for the system. Consequently, the corresponding N^{th} approximating system defined by (69)-(72) is not a minimal (i.e. not controllable and observable) and can be reduced in dimension. In particular, if we define the *REDUCED N^{th} approximating system* by the ordinary differential equation

$$E_R^N(p)\,\dot{w}^N(t) = A_R^N(p)\,w^N(t) + G_R^N u(t)\,, \tag{82}$$

with initial data

$$w^N(0) = w_0^N \tag{83}$$

and output

$$y_R^N(t) = H_R^N w^N(t)\,, \tag{84}$$

where $H_R^N = [H, 0, \ldots, 0]$ and $G_R^N = \mathrm{col}\,[G, 0, \ldots, 0]$, then one can show that (69)-(71) and (82)-(84) have the same input-output operator (i.e. $y_R^N(t) = y^N(t)$).

The dimension of the reduced system defined by (82)-(84) depends on N, n and n_d. Moreover, the system is largest when $n_d = n$. For applications to the airfoil in Figure 1.1, the value of $n_d = 1$ is reasonable and in this case the dimension of the reduced system (82)-(84) is given by $DIM(N,n) = [N + (4 + n)]$. Table 4.2 illustrates the sizes of the *REDUCED approximating systems* for various values of N and n, in the case that $n_d = 1$. If one compares Table 4.2 with Table 4.1, then it becomes clear that for large values of N and n the system (82)-(84) is much smaller (and hence much more manageable for numerical calculations) than the non-reduced AVE system (69)-(72). For example, if N = 64 and n = 3 then the dimension of (69)-(72) is 455 while the dimension of the "equivalent" system (82)-(84) is 71!

TABLE 4.2. The dimension of the REDUCED AVE approximating systems for values of N and n.

	$n_d = 1$		
N	DIMR(N,1)	DIMR(N,2)	DIMR(N,3)
2	7	8	9
4	9	10	11
8	13	14	15
16	21	22	23
32	37	38	39
64	69	70	71

Corresponding to problems (ID) and (IDN) we have;

PROBLEM (IDN_R). *Given the input function* u *and observations* $\bar{y}_i$ *at times* $\bar{t}_i$ $(0 \le \bar{t}_1 < \bar{t}_2 < \ldots < \bar{t}_m = T)$, *find the system parameters* $\hat{\gamma}_R^N = (\hat{p}_R^N, \hat{r}_R^N) \in \Gamma$ *which minimize the error*

$$J_R^N(\gamma) = \frac{1}{2} \sum_{i=1}^{M} \| y_R^N(\bar{t}_i;\gamma) - \bar{y}_i \|^2, \tag{85}$$

where $y_R^N(t;\gamma)$ *is the output at time* t *to the reduced* N^{th} *approximating system* (82)-(84).

Although we restricted ourselves to the AVE approximating scheme, the same basic ideas can be applied to the SPLINE scheme. However, there are some differences in the two algorithms. For example, it is not true that the REDUCED SPLINE SYSTEM has the same input-output operator as the SPLINE SYSTEM.

SECTION 5

NUMERICAL EXAMPLES

In this section we present some numerical examples to illustrate the method discussed above. All the numerical results were run under the Conversational Monitor System on an IBM 370/158 at the Virginia Tech Computer Center. The test model is described by the second order delay equation

$$\frac{d}{dt}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega^2 & 0 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & -a \end{bmatrix}\begin{bmatrix} x_1(t-r) \\ x_2(t-r) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u_{.1}(t), \tag{86}$$

with initial data

$$\begin{bmatrix} x_1(s) \\ x_2(s) \end{bmatrix} \equiv \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad s \leq 0 \tag{87}$$

and output

$$\begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \tag{88}$$

where $u_{.1}(t)$ is the unit step at $t = 0.1$. The final time of $T = 2$ is used in each example.

The "observations" $\bar{y}_1$ were generated by selecting the "true" set of parameters $\gamma^* = (\omega^*, a^*, r^*) = (4, 10, 1)$, using the method of steps to solve (86)-(88) for y(t) and settin~ $\bar{y}_j = y(\bar{t}_j)$. Using this procedure, we generated 101 equally spaced data points on the interval [0,2].

For various values of N = 2, 4, 8, 16, . . . the two problems PROBLEM (IDN) and PROBLEM (IDN_R) were solved. The amount of CPU time required to solve each problem was recorded in order to compare the two algorithms. In addition to the AVE scheme detailed above, we tested the SPLINE scheme so that a comparison of these two schemes could also be made. The maximum likelihood algorithm described in [8] was used in all of the calculations.

EXAMPLE 5.1

In this example we assumed that the time delay $r = 1$ is known and estimate ω and a. Since the maximum likelihood estimator is an iterative algorithm, start up values have to be provided. For this example the initial estimates were $\omega = \sqrt{15}$ and $a = 8.0$. The numerical results for this example are summarized in Tables (5.1) - (5.2).

Observe that the AVE scheme did not converge for $N = 2, 4$. For $N = 32$ the dimension of PROBLEM (IDN) is 66. Consequently, (because of storage limitations) this approximating system is "two large" and we could not solve PROBLEM (IDN) for $N = 32$. The AVE scheme produces exactly the same parameter estimates as the REDUCED AVE acheme. However, the REDUCED AVE scheme requires considerably less CPU time than the AVE scheme.

The SPLINE algorithm converged for all values of N. Moreover, as expected the SPLINE scheme produced better parameter estimates than did the AVE scheme. Note that not only does the reduced problem require less CPU time, the REDUCED SPLINE scheme produces better parameter estimates than the full SPLINE scheme. Figures 5.2 through 5.3 illustrate typical "converged" data fits for $N = 8$, 16, 32 using the AVE scheme. Figure 5.4 shows the $N = 8$ SPLINE data fits. The figures are plots constructed by using the reduced approximating systems. However, it is impossible to distinguish these plots from the plots one obtains using the nonreduced approximating systems and therefore those plots are not included.

TABLE 5.1 A comparison of parameter estimates obtained using the AVE and REDUCED AVE approximations.

PROBLEM (IDN) - AVE				PROBLEM (IDN_R) - AVE			
N	$\hat{\omega}^N$	$\hat{a}^N$	CPU	N	$\hat{\omega}_R^N$	$\hat{a}_R^N$	CPU
2	DID	NOT	CONVERGE	2	DID	NOT	CONVERGE
4	DID	NOT	CONVERGE	4	DID	NOT	CONVERGE
8	3.2476	15.5113	47 sec	8	3.2476	15.5113	20 sec
16	3.6431	12.2373	171 sec	16	3.6431	12.2373	47 sec
32	—	—	—	32	3.8233	11.1275	189 sec
True	4.0000	10.0000		True	4.0000	10.0000	

TABLE 5.2 A comparision of the parameter estimates obtained using the SPLINE and REDUCED SPLINE approximations.

PROBLEM (IDN) - SPLINE				PROBLEM (IDN_R) - SPLINE			
N	$\hat{\omega}^N$	$\hat{a}^N$	CPU	N	$\hat{\omega}_R^N$	$\hat{a}_R^N$	CPU
2	3.7738	7.8675	18	2	3.6466	9.1217	8
4	3.9674	9.9133	17 sec	4	3.9535	10.3377	13 sec
8	3.9981	9.9651	36 sec	8	3.9950	10.0643	20 sec
16	3.9979	9.9731	94 sec	16	3.9970	10.0000	41 sec
32	—	—	—	32	3.9976	9.9842	184 sec
True	4.0000	10.0000		True	4.0000	10.0000	

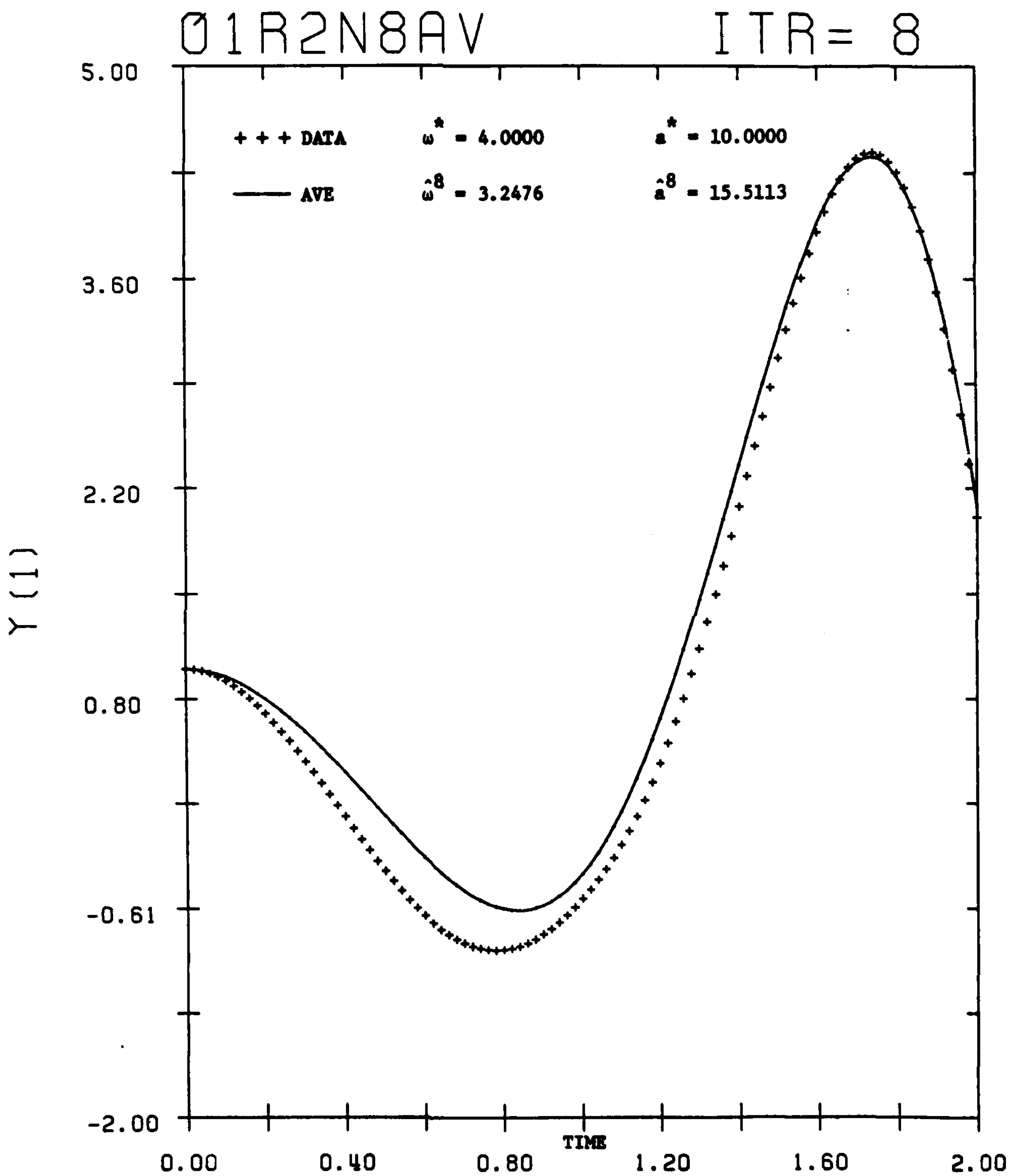

FIGURE 5.1. The data fits for Example 5.1 using N = 8 and the AVE approximations.

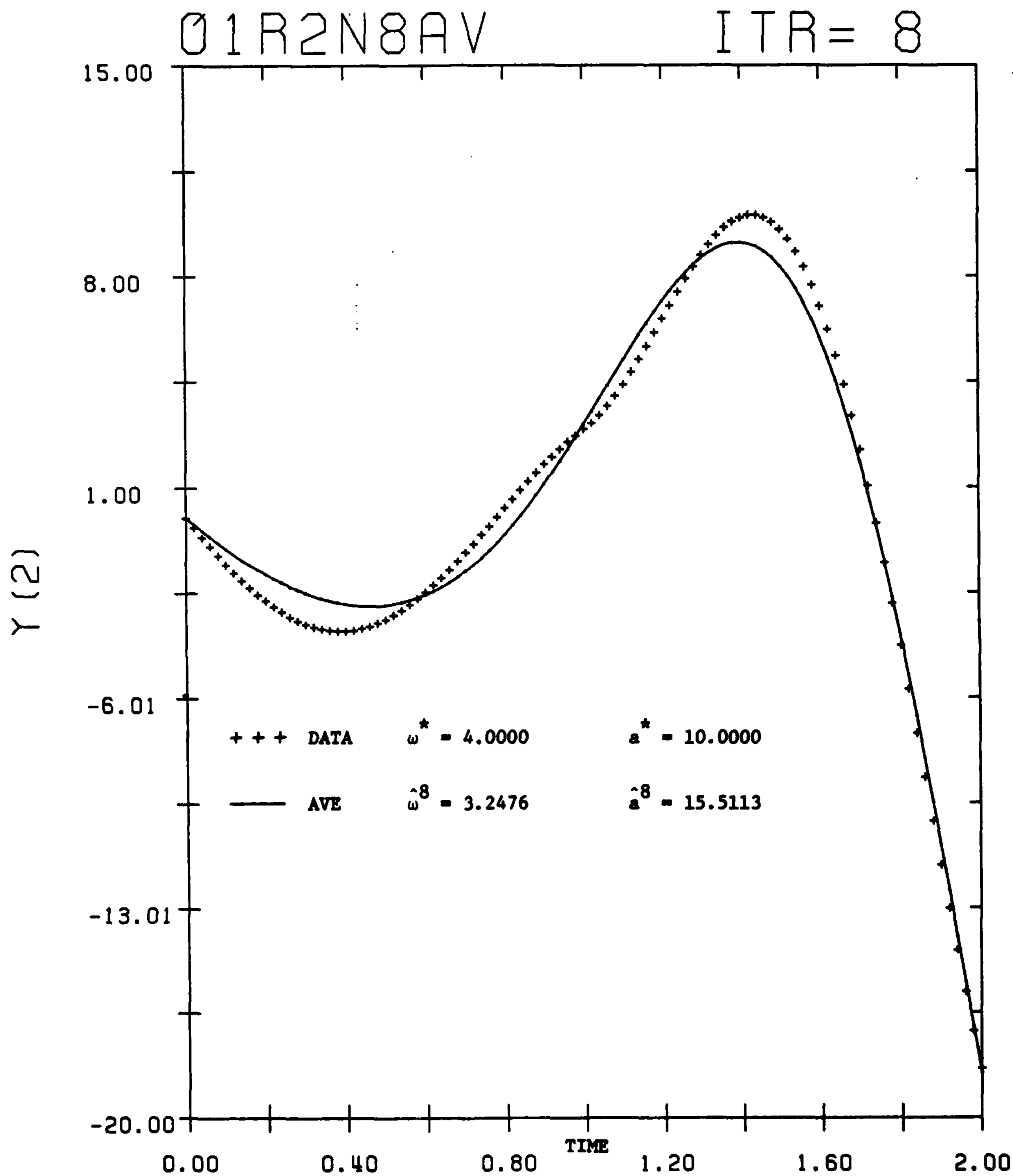

FIGURE 5.1.b. The data fits for Example 5.1 using N = 8 and the AVE approximations.

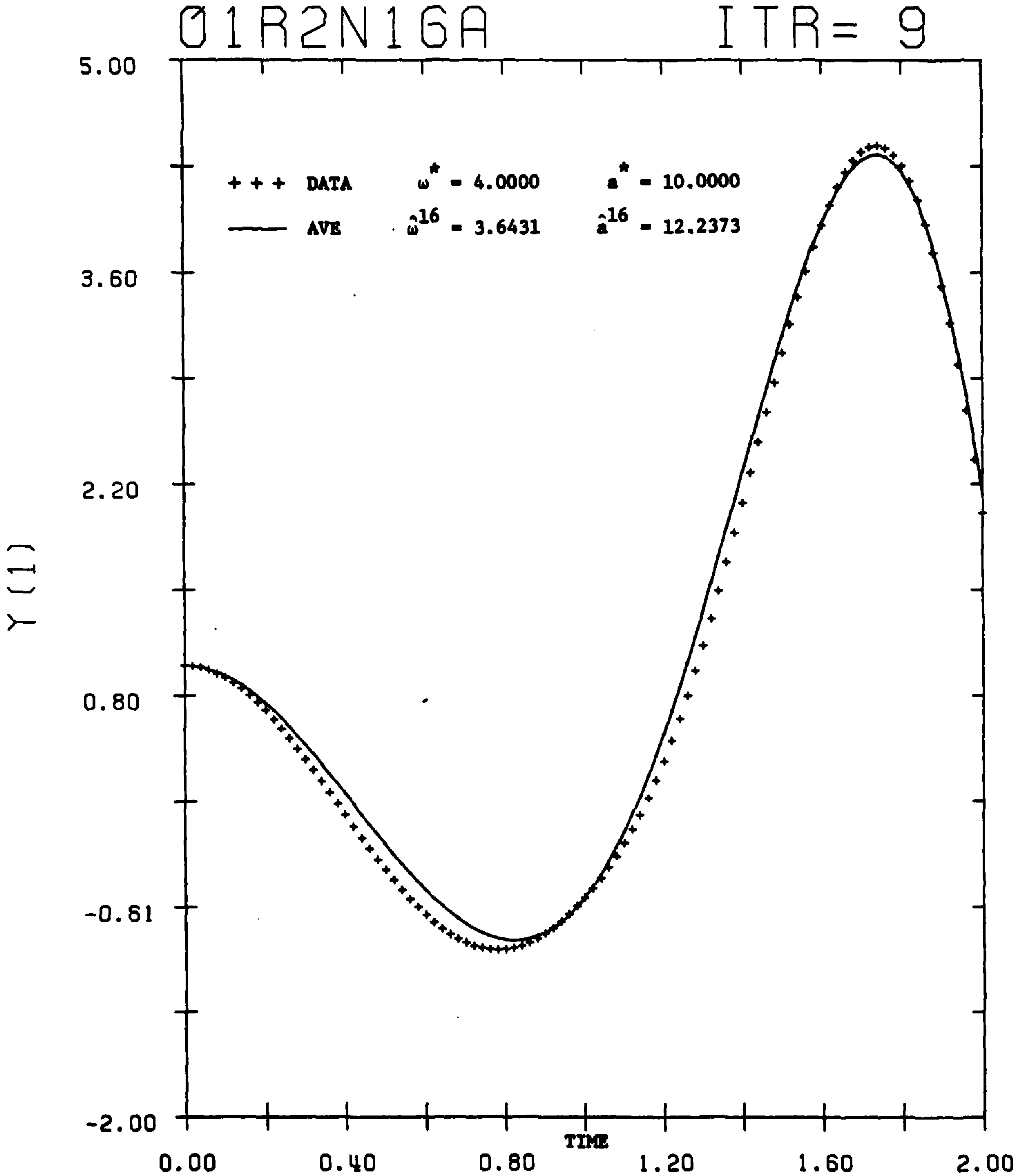

FIGURE 5.2. The data fits for Example 5.1 using N = 16 and the AVE approximations.

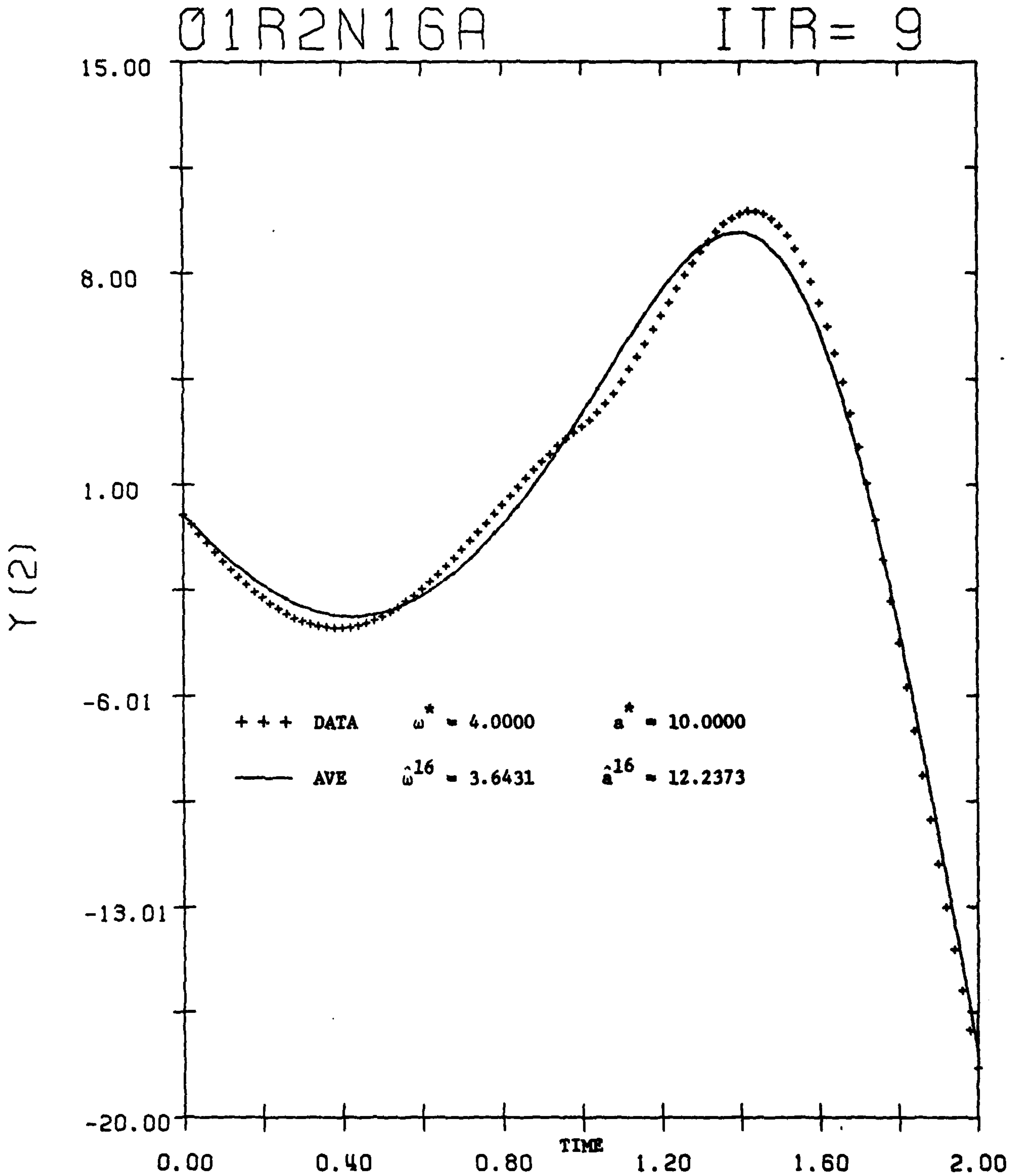

FIGURE 5.2.b. The data fits for Example 5.1 using N = 16 and the AVE approximations.

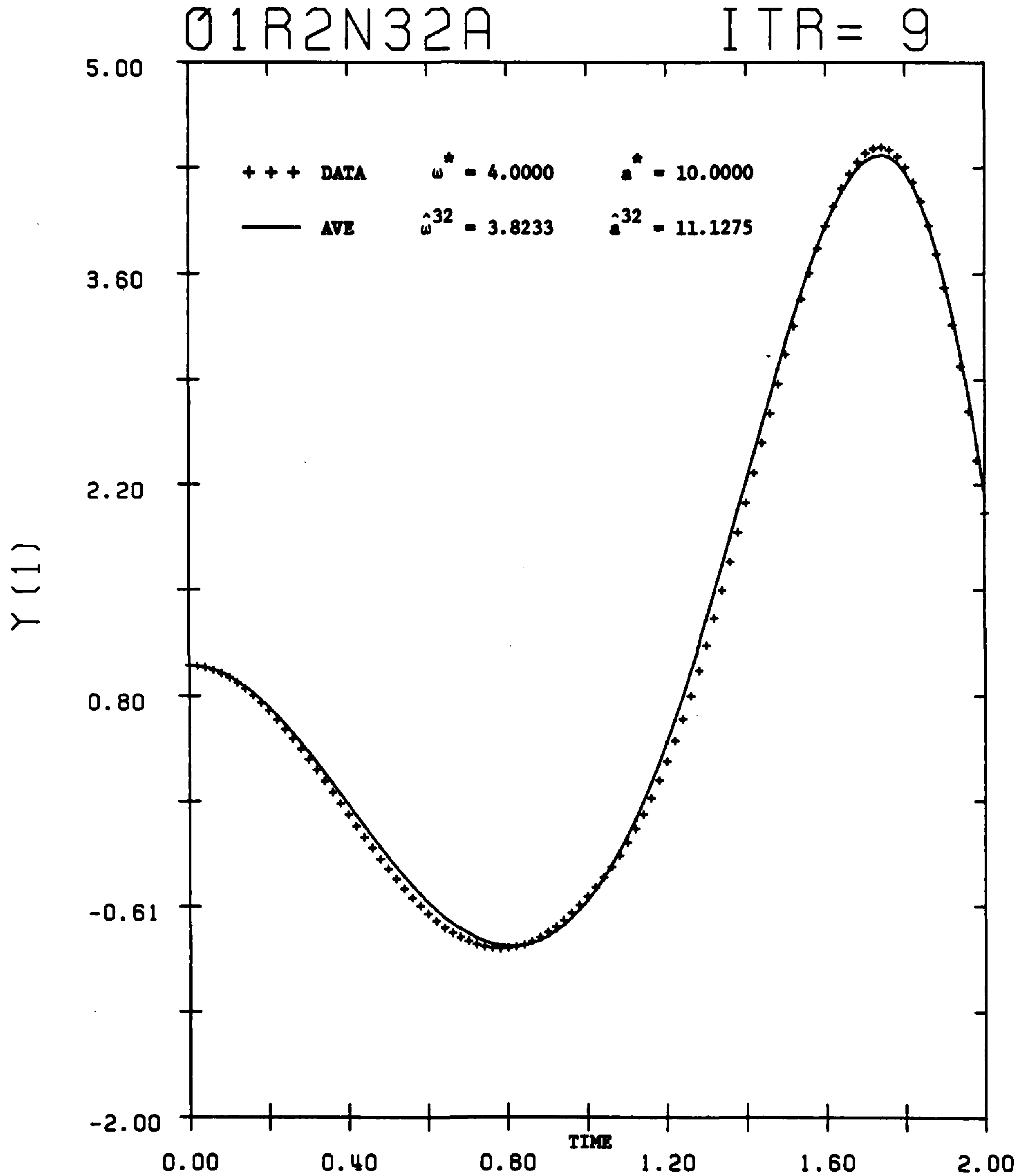

FIGURE 5.3. The data fits for Example 5.1 using N = 32 and the AVE approximations.

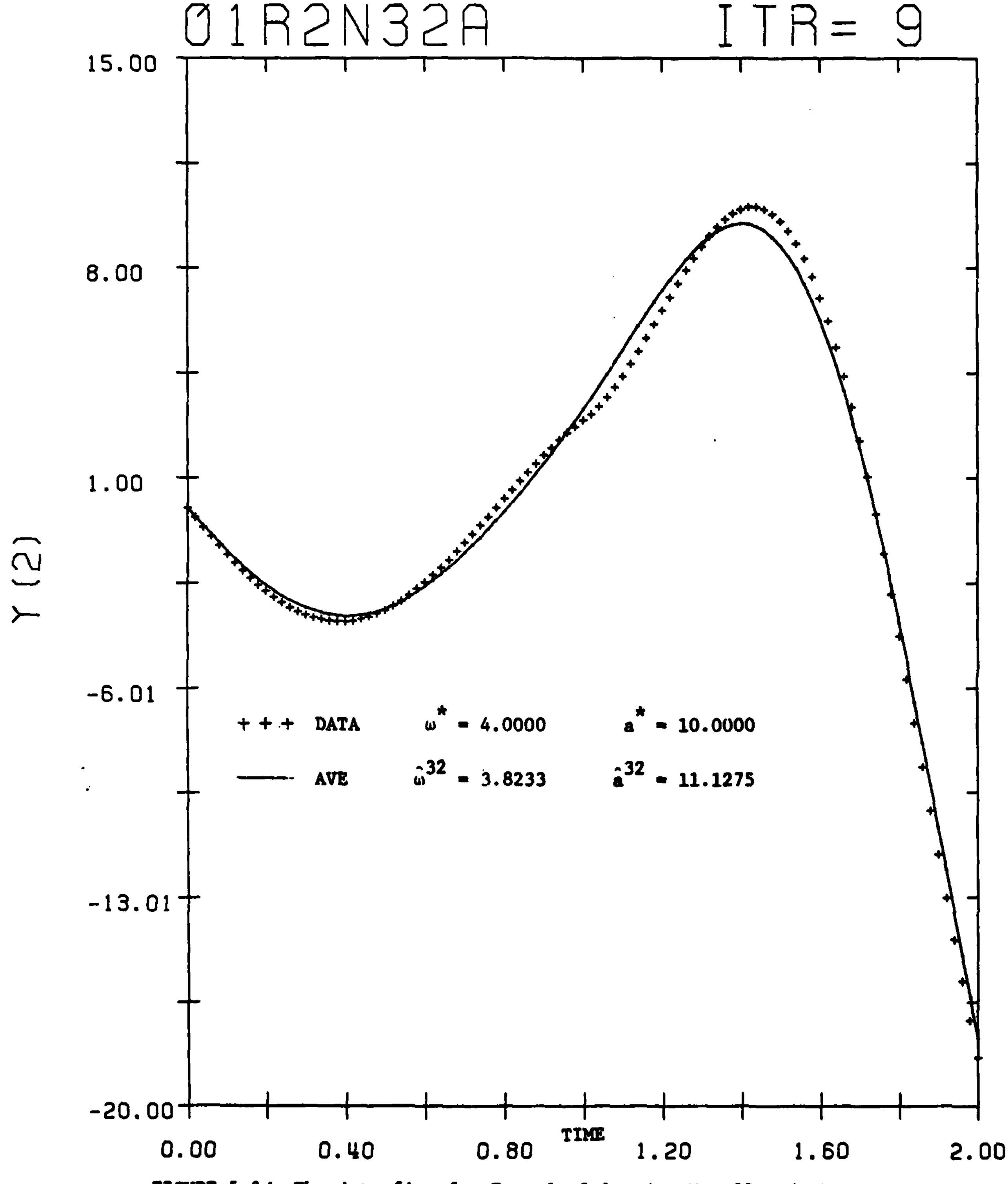

FIGURE 5.3.b. The data fits for Example 5.1 using N = 32 and the AVE approximations.

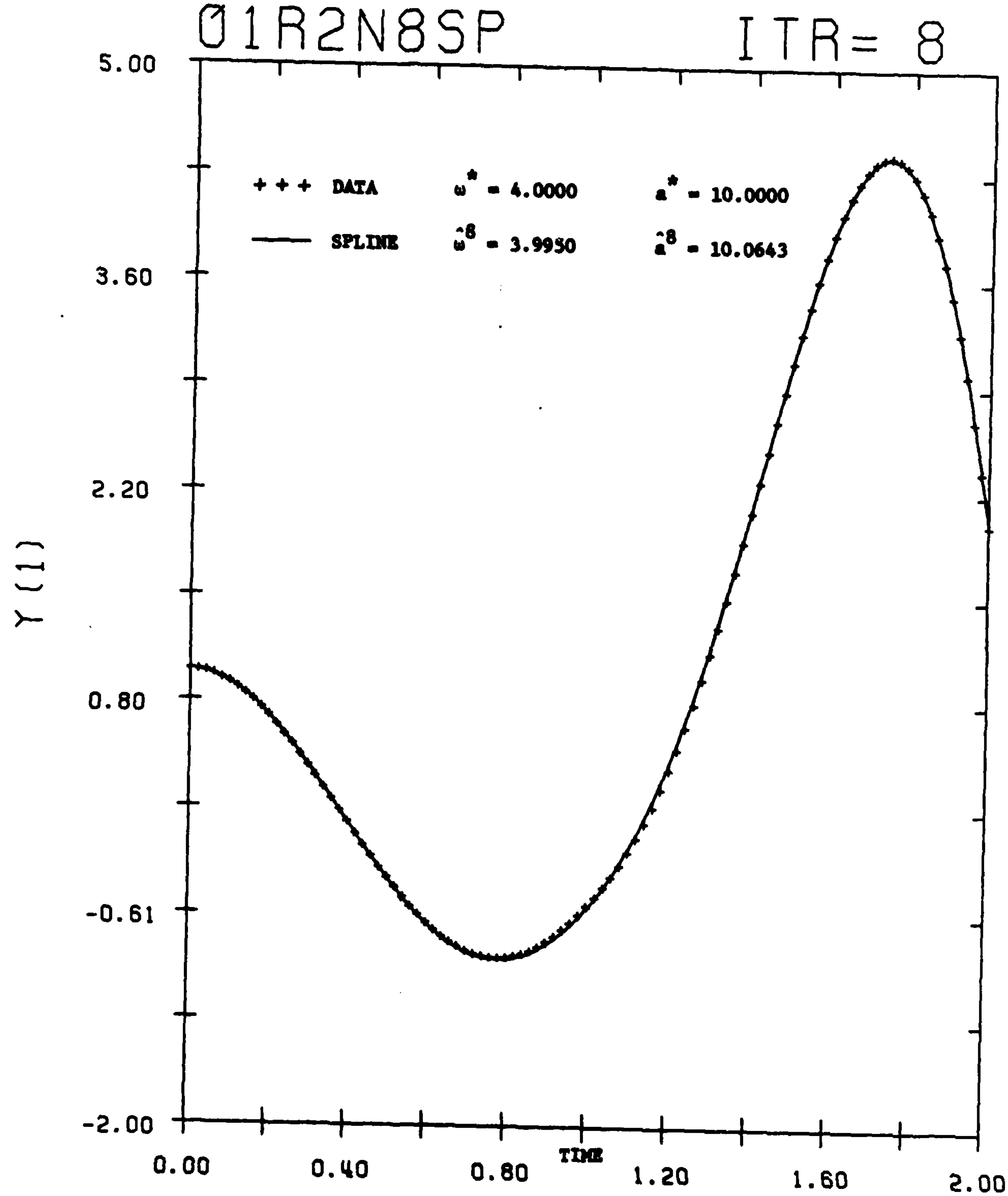

FIGURE 5.4. The data fits for Example 5.1 using $N = 8$ and the SPLINE approximations.

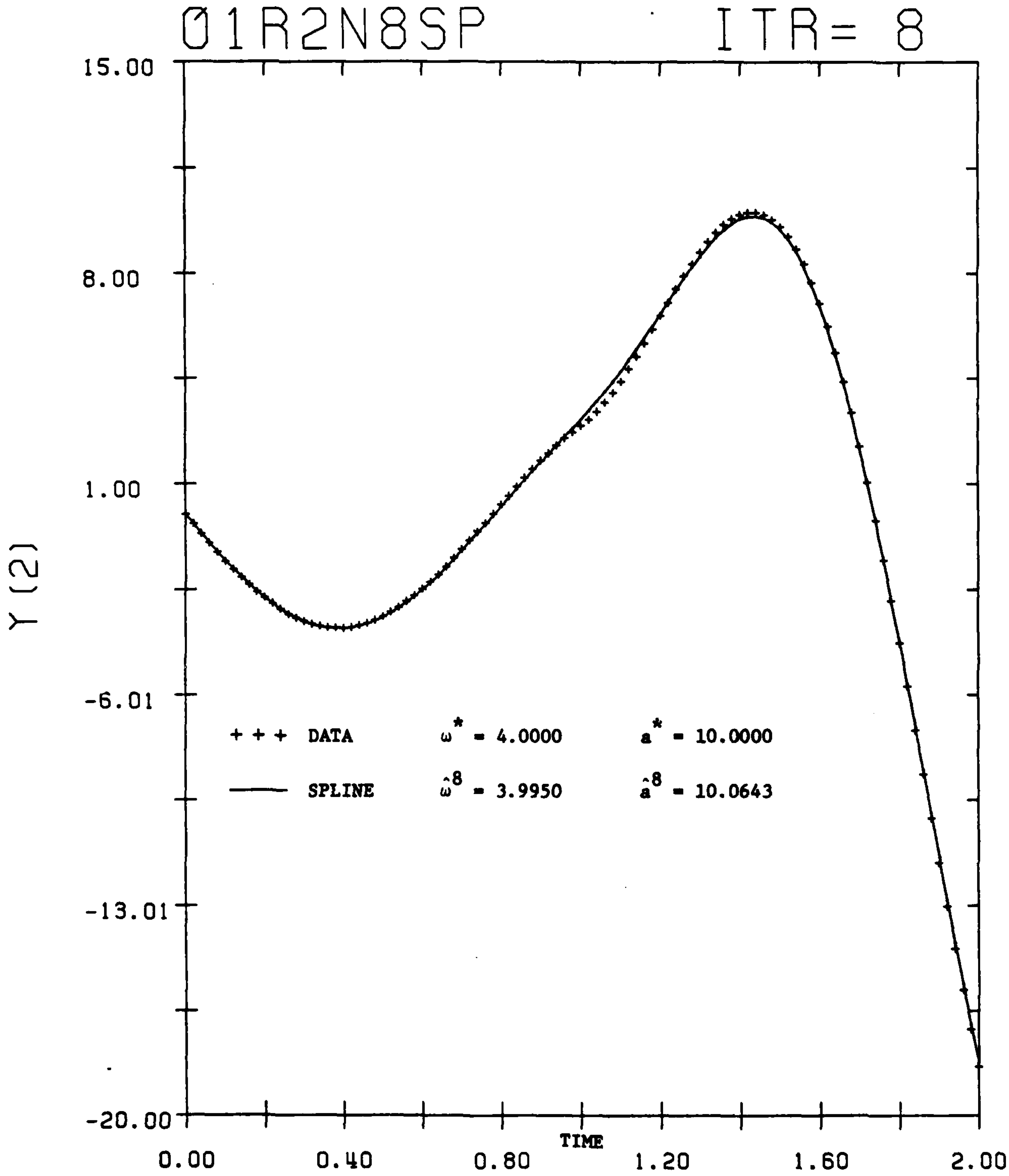

FIGURE 5.4.b. The data fits for Example 5.1 using N = 8 and the SPLINE approximations.

EXAMPLE 5.2.

We now consider the problem of estimating all three parameters (ω, a, r) in the model (86)-(88). Start up values were taken to be $\omega = \sqrt{15}$, a = 8.0 and r = 0.8. The addition of the time delay as an unknown parameter increases the complexity of the problem and the AVE scheme never converged. However, the SPLINE scheme converged for all admissible values of N. Tables 5.3 and 5.4 contain a summary of the numerical results for this example. As in the previous example, the reduced problem required less CPU time to converge and generally produced better parameter estimates than the full SPLINE scheme. Figures 5.5 through 5.7 illustrate the converged data fits for N = 2, 8, 32.

TABLE 5.3. Example 5.2 parameter estimates using SPLINE.

SPLINE SCHEME - PROBLEM (IDN)				
$\underline{N}$	$\underline{\hat{\omega}^N}$	$\underline{\hat{a}^N}$	$\underline{\hat{r}^N}$	$\underline{\text{CPU}}$
2	3.4340	8.1849	.8725	31 sec
4	3.8313	9.0540	.9476	28 sec
8	3.9949	9.9439	.9989	60 sec
16	3.9986	9.9775	1.0002	241 sec
True	4.0000	10.0000	1.0000	

TABLE 5.4. Example 5.2 parameter using REDUCED SPLINE.

SPLINE SCHEME - PROBLEM (IDN_R)				
$\underline{N}$	$\underline{\hat{\omega}^N_R}$	$\underline{\hat{a}^N_R}$	$\underline{\hat{r}^N_R}$	$\underline{\text{CPU}}$
2	3.5183	9.0515	.9445	15 sec
4	3.8092	9.3691	.9457	19 sec
8	3.9919	10.0448	.9991	39 sec
16	3.9977	10.0002	1.0002	104 sec
32	3.9979	9.9862	1.0001	436 sec
True	4.0000	10.0000	1.0000	

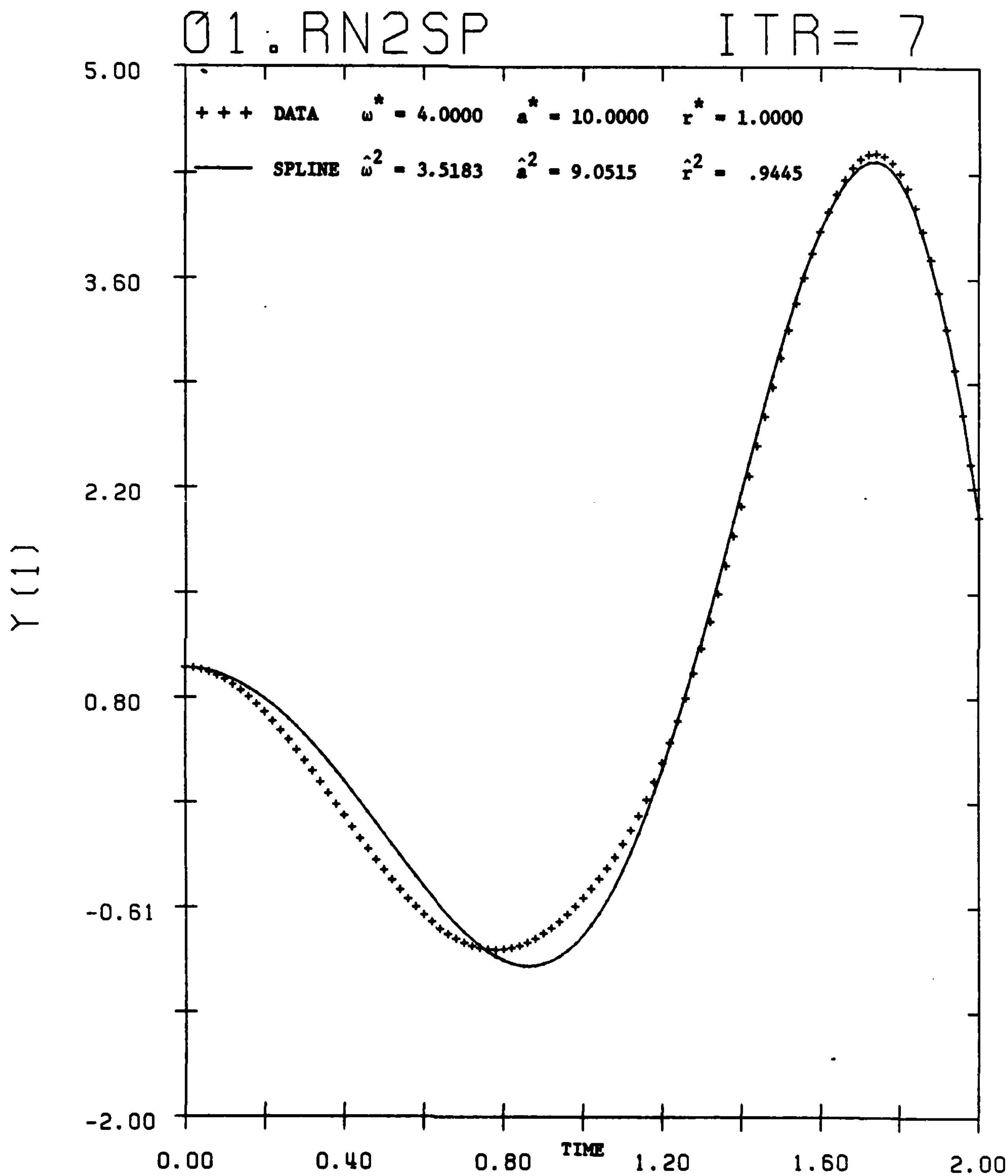

FIGURE 5.5. The data fits for Example 5.2 using N = 2 and the SPLINE approximations.

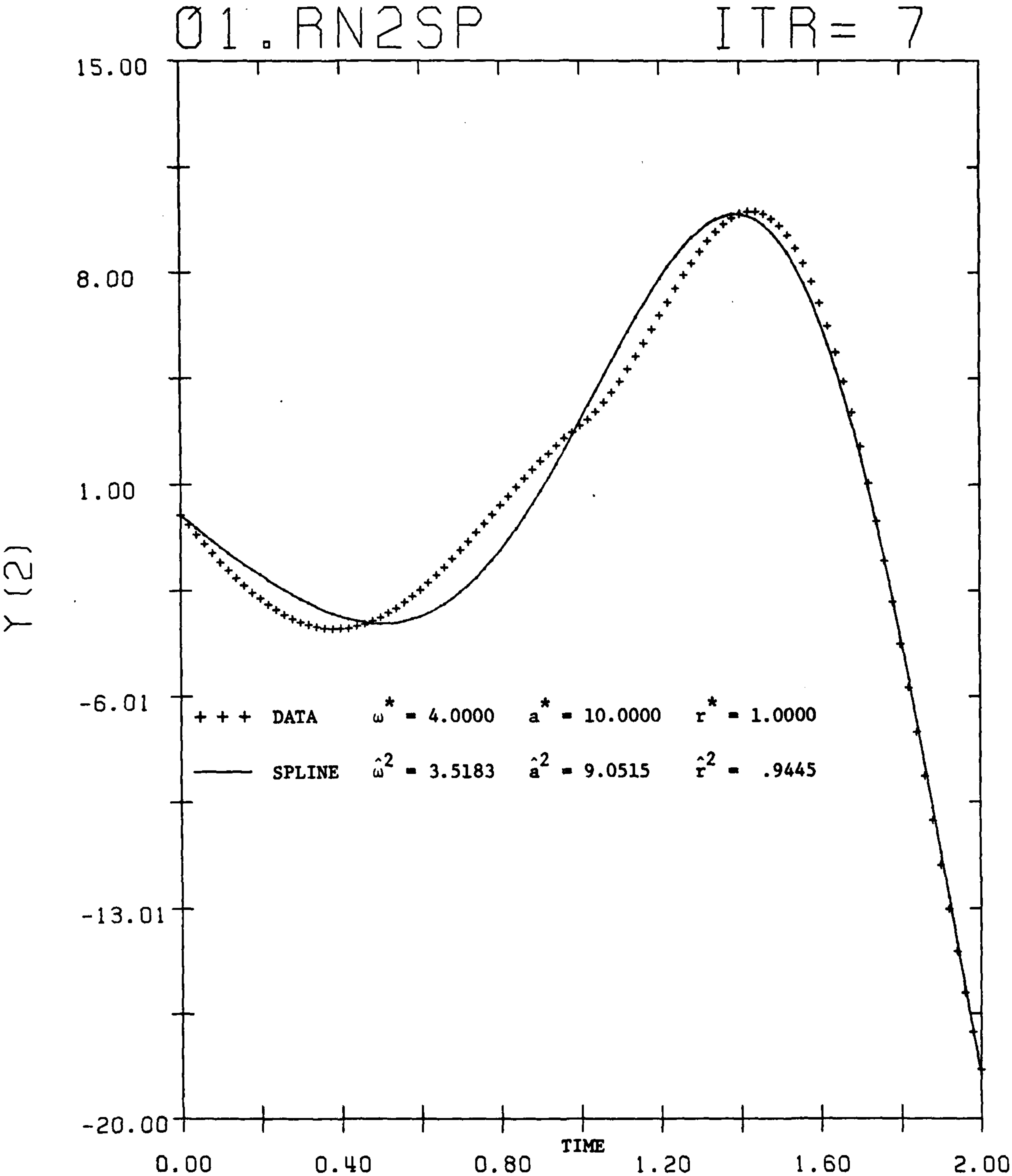

FIGURE 5.5.b. The data fits for Example 5.2 using N = 2 and the SPLINE approximations.

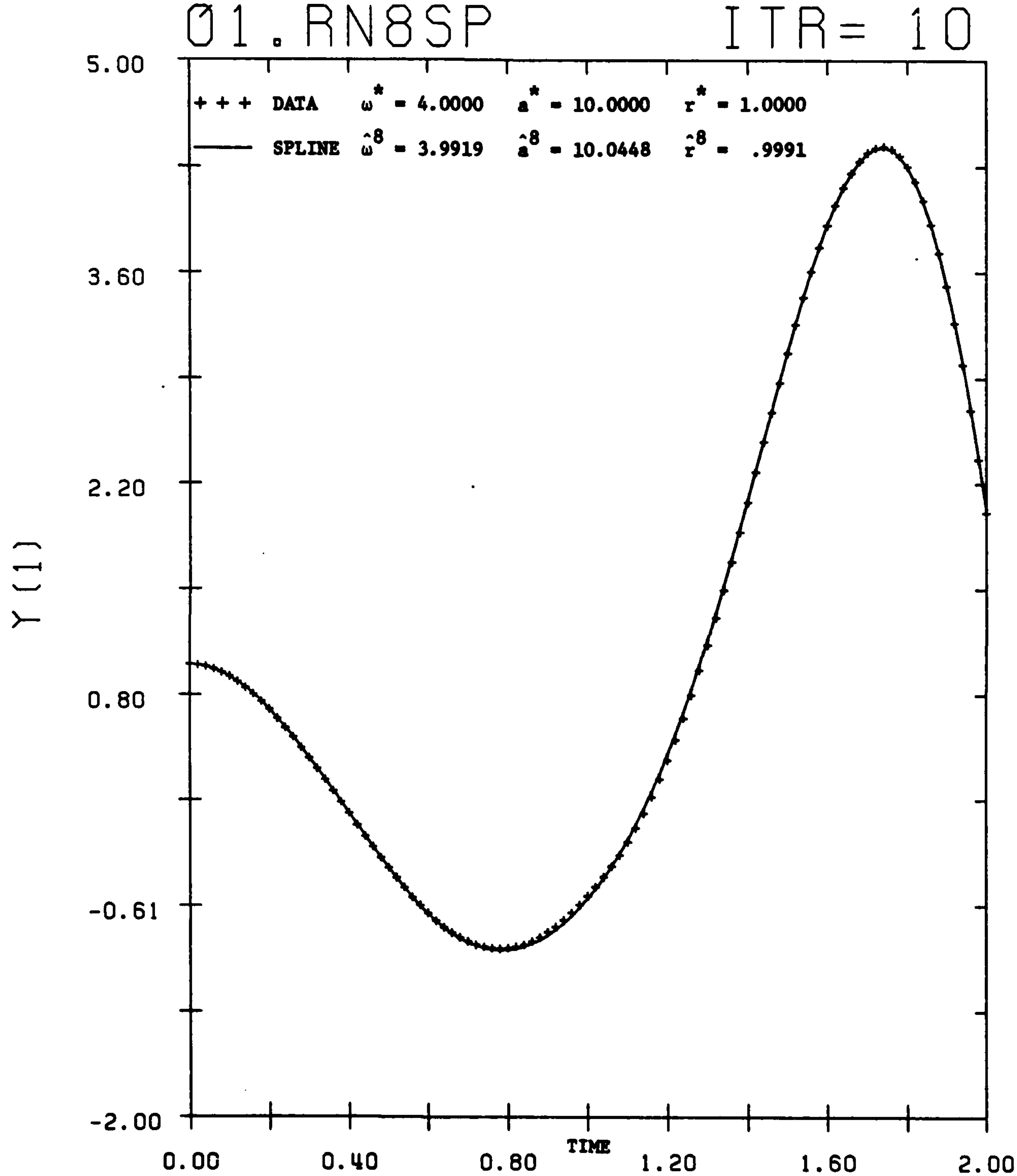

FIGURE 5.6. The data fits for Example 5.2 using N = 8 and the SPLINE approximations.

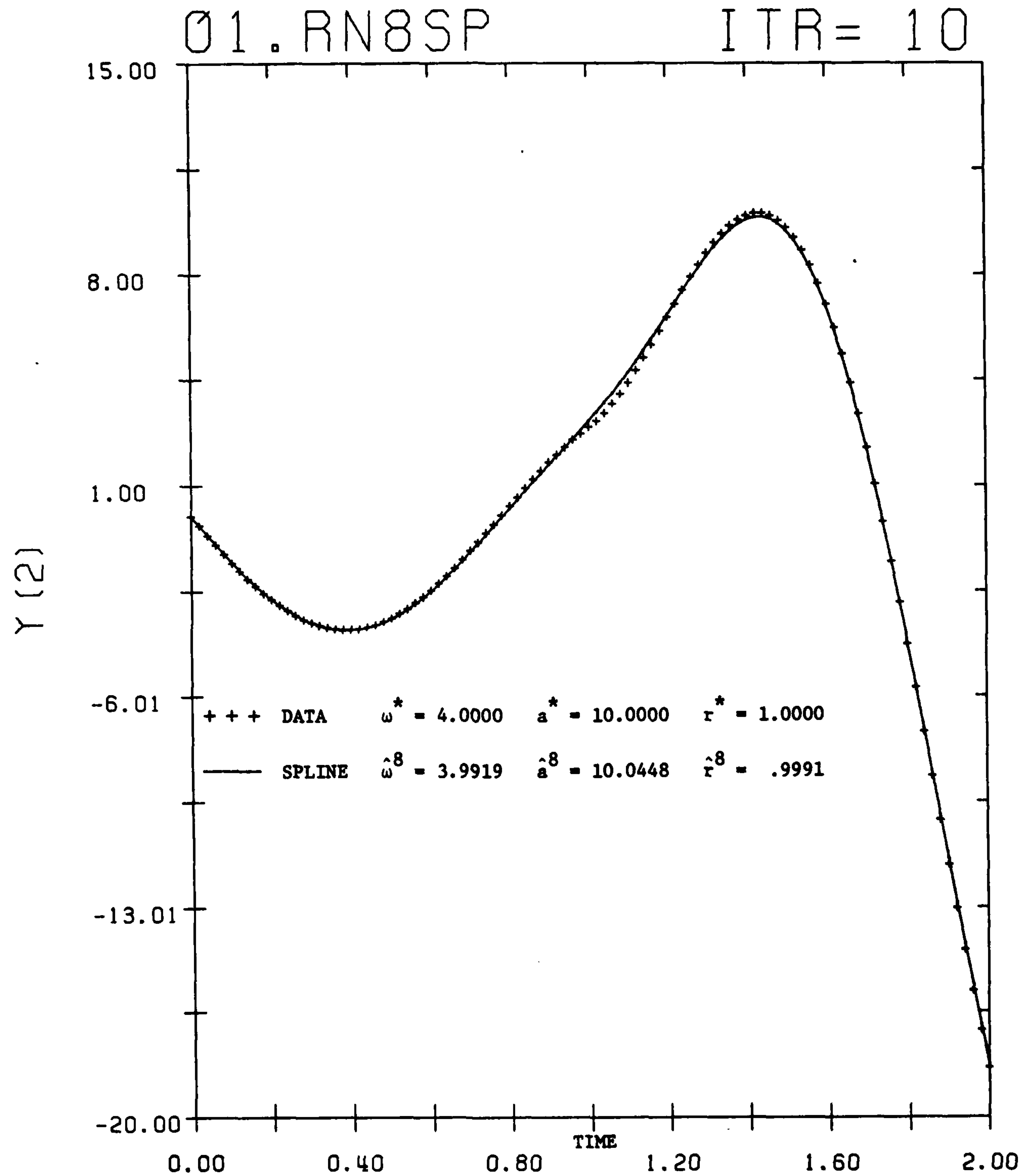

FIGURE 5.6.b. The data fits for Example 5.2 using N = 8 and the SPLINE approximations.

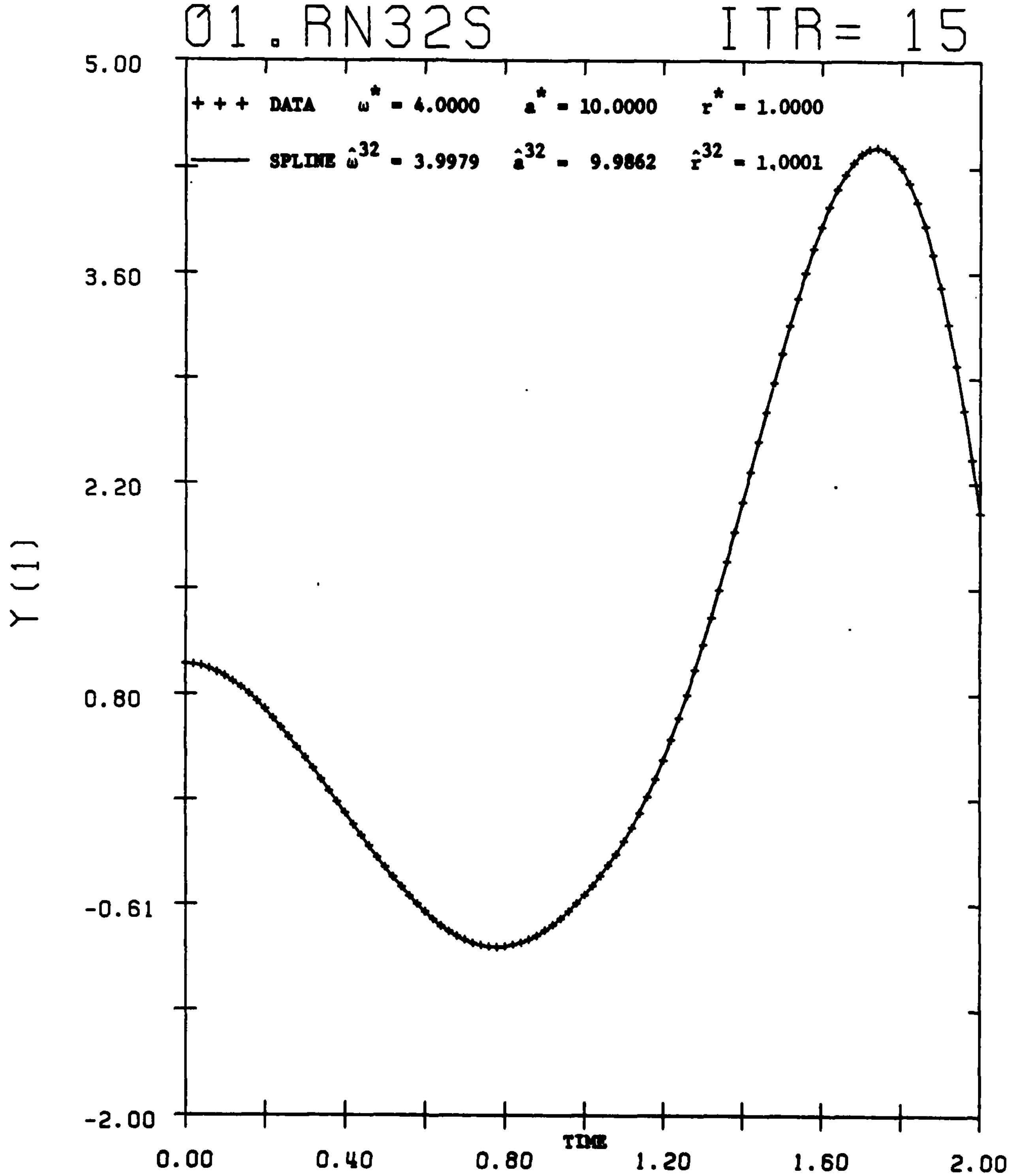

FIGURE 5.7. The data fits for Example 5.2 using $N = 32$ and the SPLINE approximations.

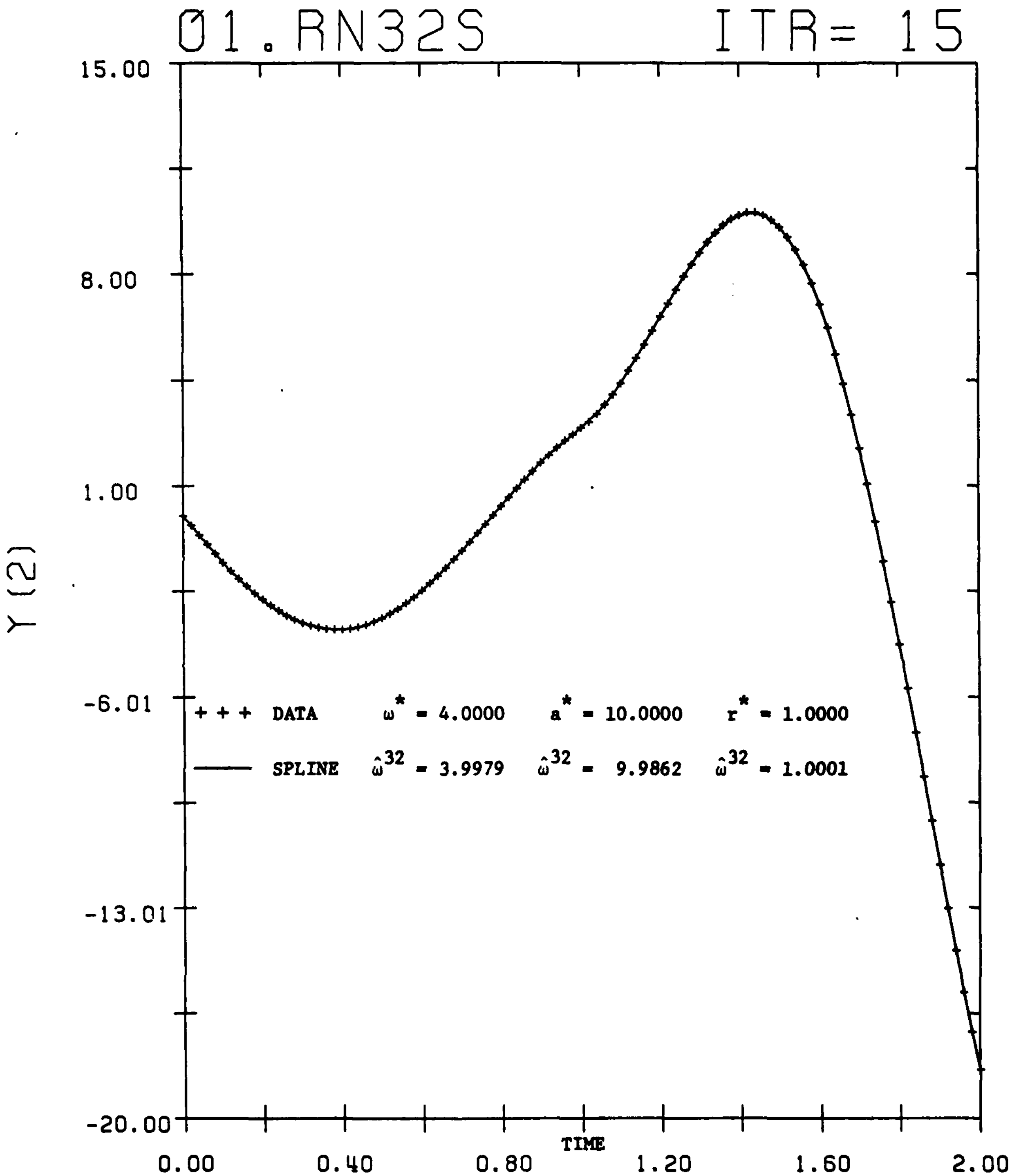

FIGURE 5.7.b. The data fits for Example 5.2 using N = 32 and the SPLINE approximations.

SECTION 6

PRELIMINARY ANALYSIS OF WIND-TUNNEL DATA

Several numerical experiments were conducted using actual wind-tunnel data for a system similar to that shown in Figure 1.1. The experiments were performed at Neilsen Engineering and Research, Inc. in Mountain View, CA, under the supervision of Dr. S. C. McIntosh. Dr. McIntosh gratiously made some of the data available to the authors.

The mathematical model appropriate for the experimental system differs slightly from that analyzed above. Firstly, the elastic axis, at which point the plunging motion is measured, is not at the quarter-chord point ($x_\alpha \neq 1/2$). Thus, while the mass matrix M_s has the form

$$M_s = \begin{bmatrix} m & S_\alpha \\ S_\alpha & i_\alpha \end{bmatrix}$$

the aerodynamic added mass-matrix is somewhat more general than the form used in (3.6)

$$\pi\rho b^2 \begin{bmatrix} 1 & b \cdot x_\alpha \\ b \cdot x_\alpha & b^2(1/8 + x_\alpha^2) \end{bmatrix}.$$

A second generalization is the inclusion of viscous damping terms C_h and C_α along the diagonal in the submatrix A_{11} (see (35)). The structural stiffness matrix is diagonal with elements k_h and k_α, as in [17].

Since the elastic axis is not necessarily at the quarter-chord, the aerodynamic moment must include a term involving the product of the lift and a moment arm $b(x_\alpha - 1/2)$. The aerodynamic variable now "feeds into" the pitch, as well as the plunging equation of motion. Thus the A_{13} sub-matrix is written (see (37)).

$$A_{13} = 2\pi \rho U b \begin{bmatrix} 1 \\ \\ b(x_\alpha - 1/2) \end{bmatrix} C.$$

The generalization to non-coincident elastic axis and aerodynamic center also requires alteration of the E_{31} sub-matrix which now has the form (see (40))

$$E_{31} = - B[1, b(1/2 + x_\alpha)] \quad .$$

The nominal values of the various system parameters are given in Table 6.1

TABLE 6.1. Nominal values of the system parameters.

parameter	value
m	$1.628 \cdot 10^{-3}$ lb-sec^2/in^2
i_α	$5.426 \cdot 10^{-3}$ lb-sec^2
S_α	$6.741 \cdot 10^{-4}$ lb-sec^2/in
b	4.95 in
x_α	.297
ρ	1.47×10^{-7} lb-sec^2/in^4
k_h	6.944 lbs/in^2
k_α	43.1 lbs
C_h	$1.9 \cdot 10^{-3}$ lb-sec/in^2
C_α	$8.7 \cdot 10^{-3}$ lb-sec
U	1325 in in/sec.

Note that the only model parameters for which values have not been specified are the matrices B and C and the matrix-valued function G(s;p). These quantities describe the dynamic model for the unsteady aerodynamic behavior (see (21)-(23)).

EXAMPLE 6.1

The first example for which we present results employs a first-order ordinary differential equation model for the unsteady aerodynamic behavior. This amounts to using a single exponential to approximate the *weighting pattern* $W(t)$ in the Wagner function representation 2.10. Thus, we take $n = 1$, $B = 1$ and $G(s;p) = g_1 u_0(s)$, where u_0 is the usual unit step at zero. In this case the Riemann-Stieltjes integral in (42) becomes

$$\int_{-r}^{0} ds[g_1 u_0(s)] x_d(t+s) = g_1 x_d(t) \ ,$$

so that (42) is an ordinary differential equation. Note that from (20) theoretically one should expect $C = 1/2$. With this information the overall system (43) is a fifth-order ordinary differential equation.

The data used consists of sampled values of the variables $\dot{h}(t)$, $\dot{\alpha}(t)$, $h(t)$ and $\alpha(t)$ at 157 equally spaced points on the interval [0, .25 sec.]. Since the data seemed relatively free of noise, the initial values for the first four components of the state were taken as the recorded values at the initial time. No initial value is available for state component x_5, the aerodynamic variable, thus it was identified. Note from (35) that the second matrix in the sum for A_{11} includes a multiplicative factor, 2π, which is theoretical lift-curve slope. In this experiment the lift-curve slope was identified so that the (1,1) and (1,2) elements in the sub-matrix A_{11} are allowed to vary accordingly. The other parameters identified are the scalar coefficient C, which appears in the sub-matrix A_{13}, and the aerodynamic time constant g_1.

Results of this experiment are illustrated in Table 6.2 and in Figures 6.1 and 6.2. In these figures $y(1) = \dot{h}$, $y(2) = \dot{\alpha}$, $y(3) = h$ and $y(4) = \alpha$.

TABLE 6.2. Parameter estimates for the ordinary differenital equation model; Example 6.1.

iteration	$x_5(0)$	C_{ℓ_α}	C	g_1
0	0.0	6.28	.500	-12.2
1	25.8	11.65	.838	-28.6
5	95.4	8.41	.724	-95.2
10	84.3	8.68	.652	-75.4
20	78.9	8.98	.628	-64.0

Shown in Figure 6.1 are the data matches using the start-up estimates for the parameters. As is evident from the y(1) and y(2) matches, the fundamental frequency is nearly correct but the actual plunge damping is higher than predicted by the model while the pitch damping is lighter. The actual data exhibits some high frequency oscillations that are completely missing in the model. The "converged" results shown in Figure 6.2 indicate some improvement in the data fits but the quality of the fit is still quite poor. It seems likely that some of the "fixed parameters" (e.g. the plunge damping parameter, C_h) may be in error. Alternatively, it may be that the assumed model structure (aerodynamics via a first-order ordinary differential equation) is not appropriate. In any case a more systematic investigation is needed. As in the previous figures we use " + + + " to denote the wind-tunnel data values and a solid line to indicate the values of the output from the mathematical models.

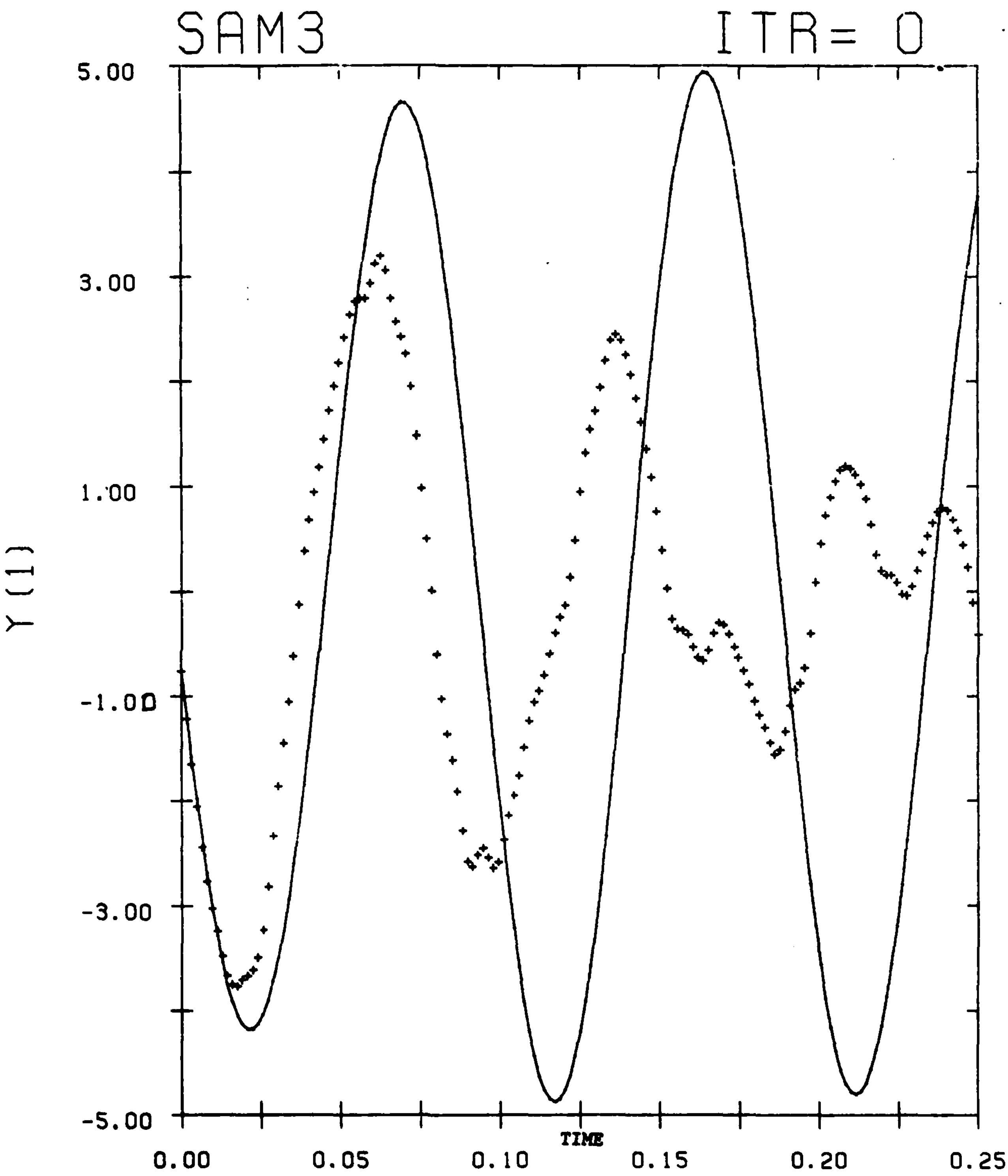

FIGURE 6.1. **The startup data fits for the ordinary differential equation model; Example 6.1.**

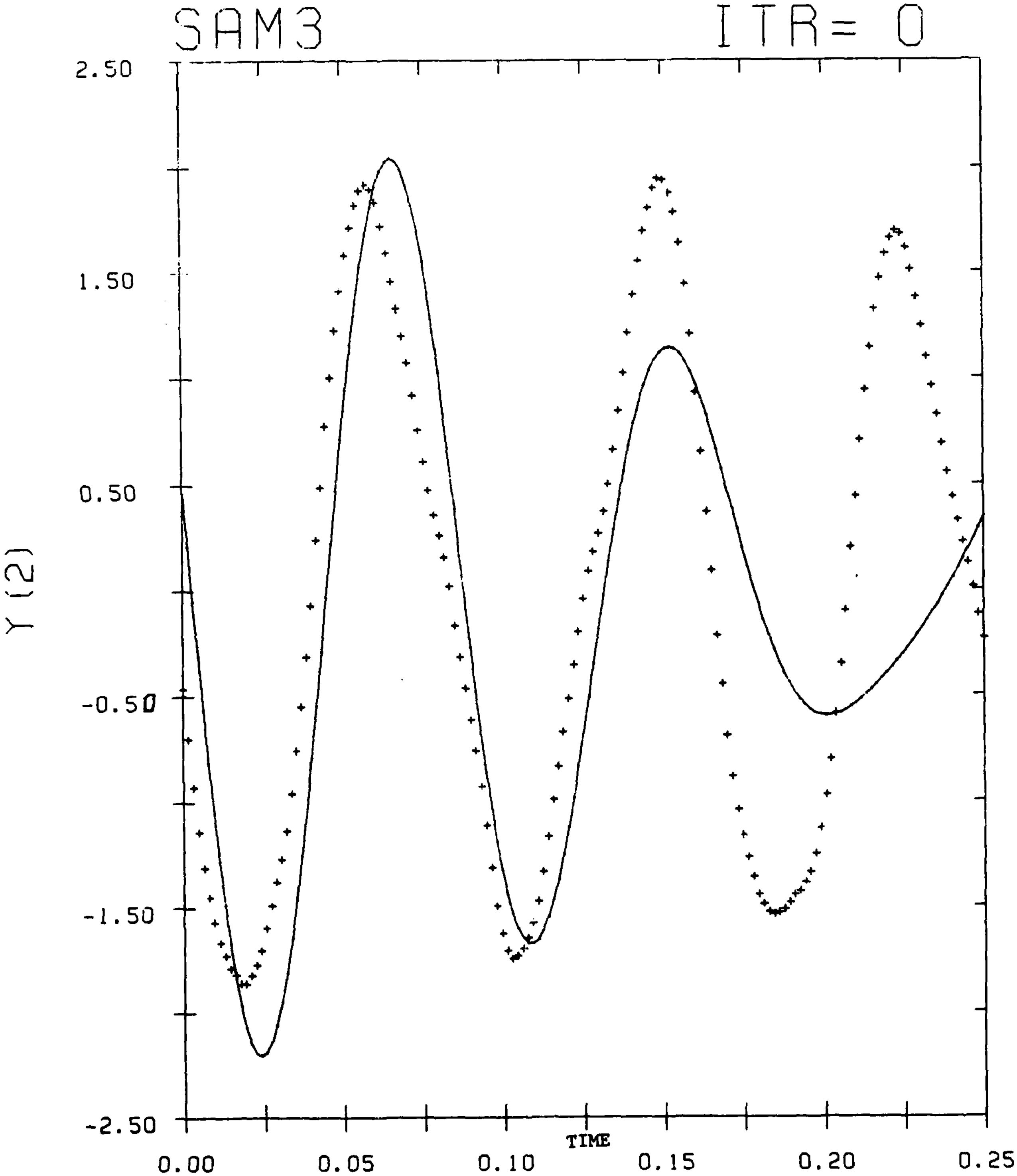

FIGURE 6.1.b. The startup data fits for the ordinary differential equation model: Example 6.1.

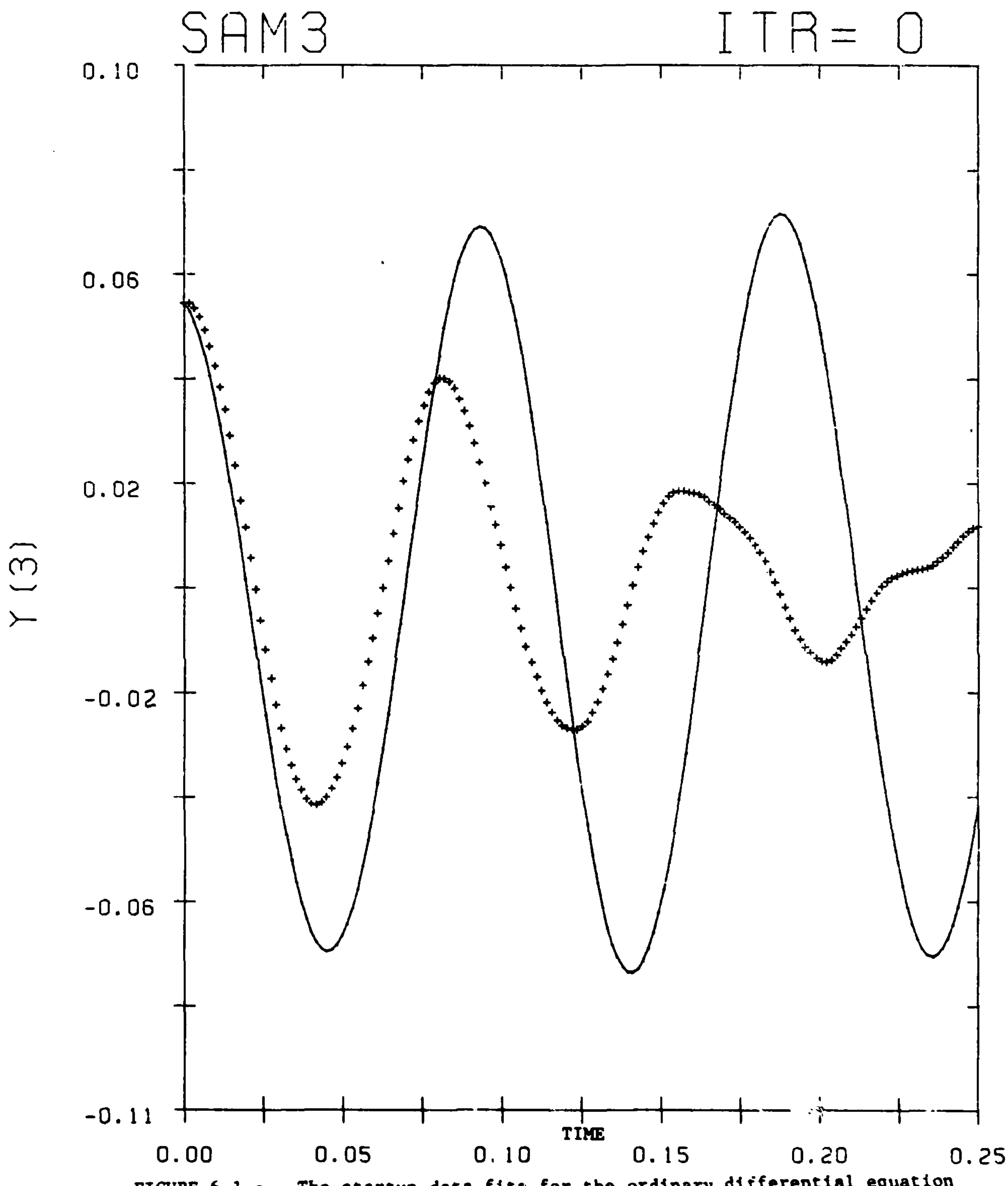

FIGURE 6.1.c. The startup data fits for the ordinary differential equation model; Example 6.1.

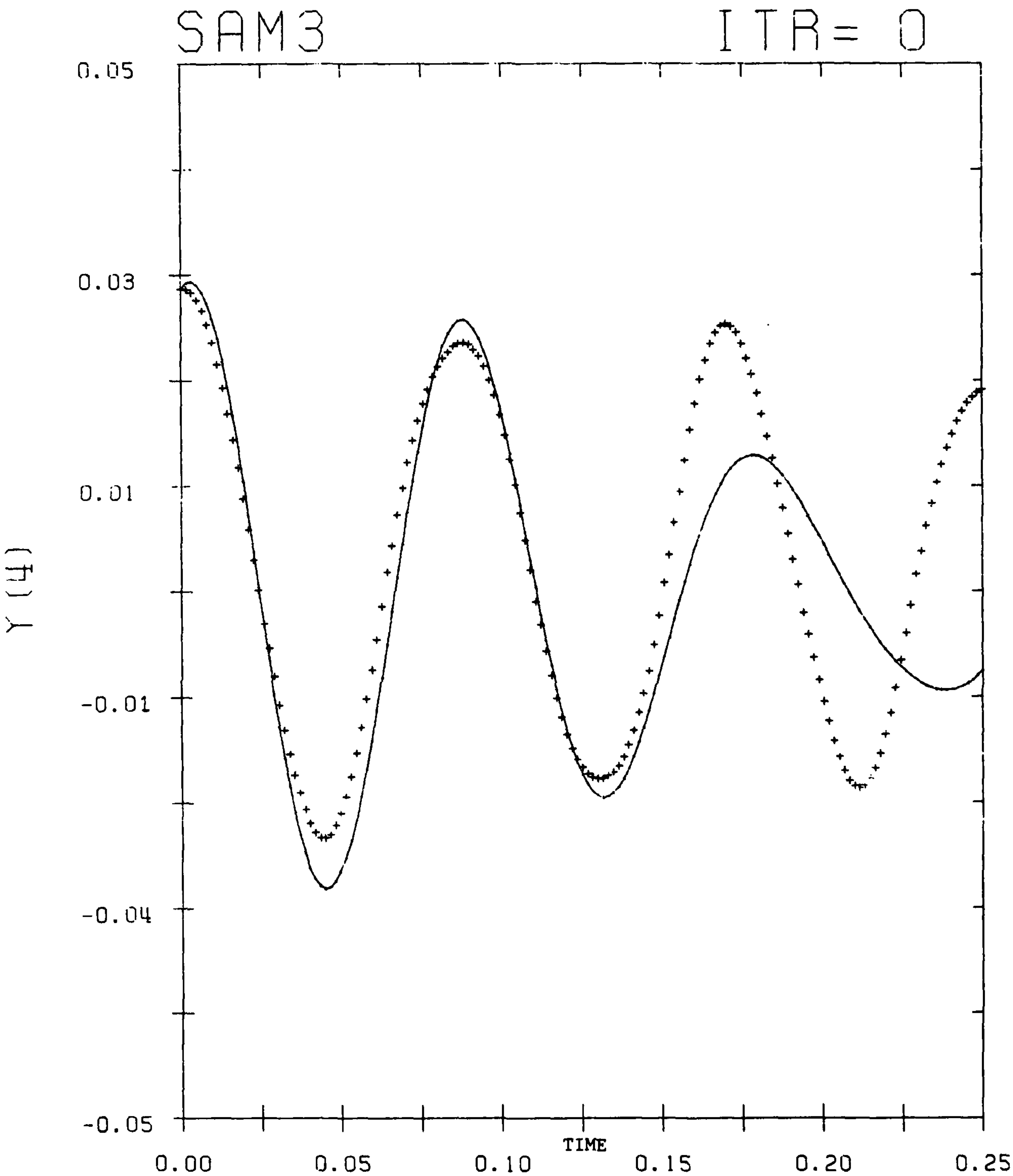

FIGURE 6.1.d. The startup data fits for the ordinary differential equation model; Example 6.1.

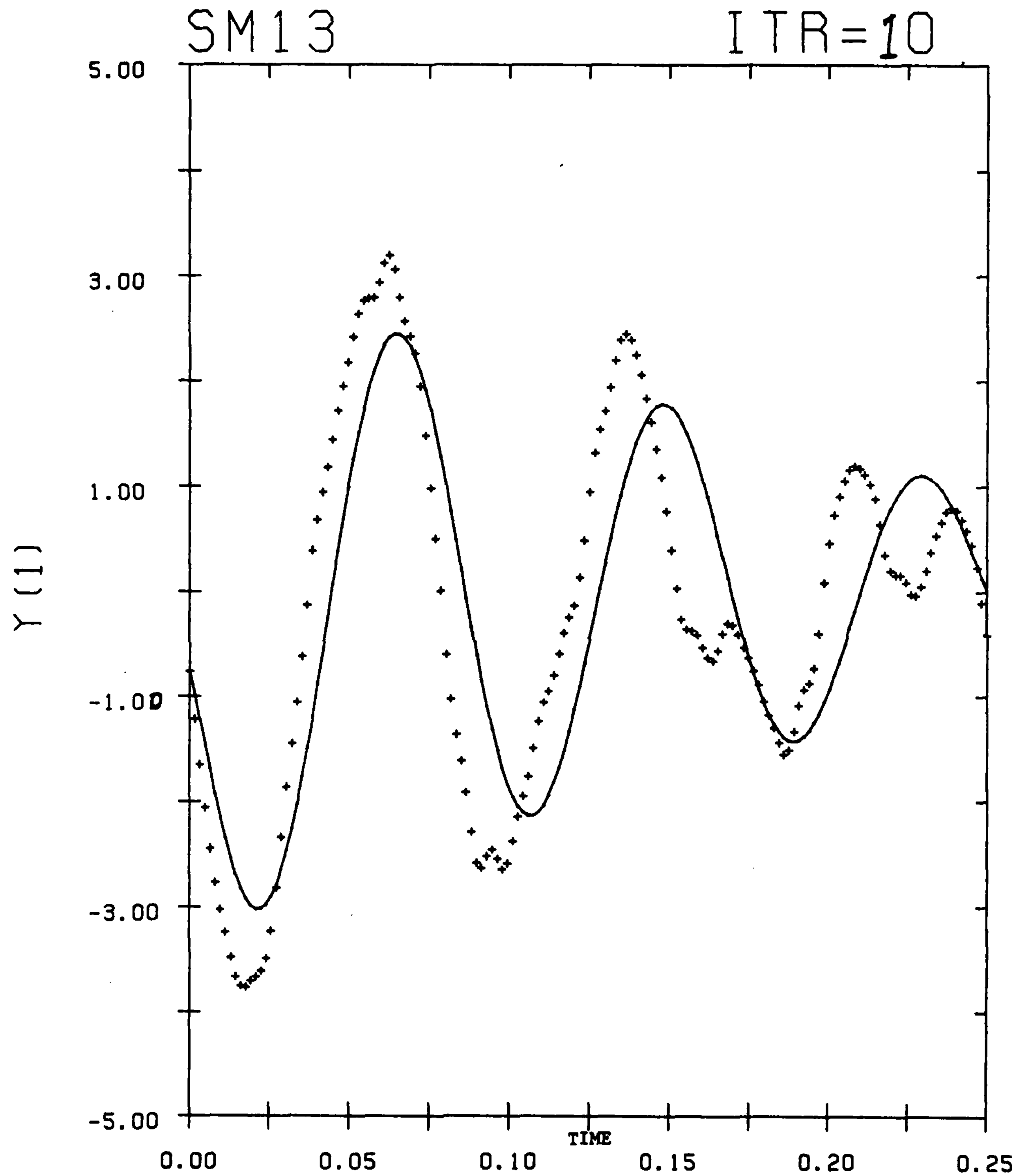

FIGURE 6.2. The converged data fits for the ordinary differential equation model; Example 6.1.

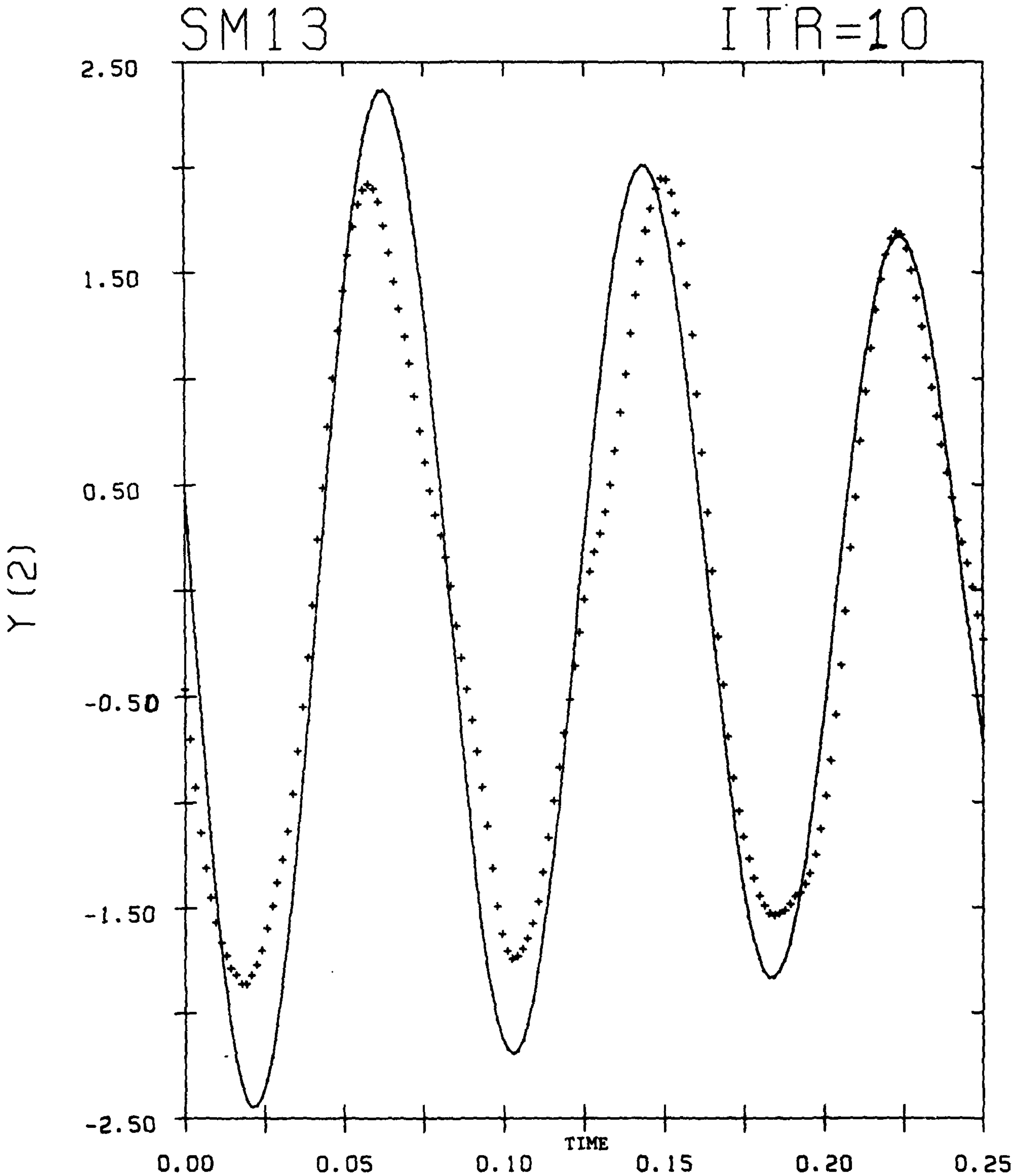

FIGURE 6.2.b. The converged data fits for the ordinary differential equation model; Example 6.1.

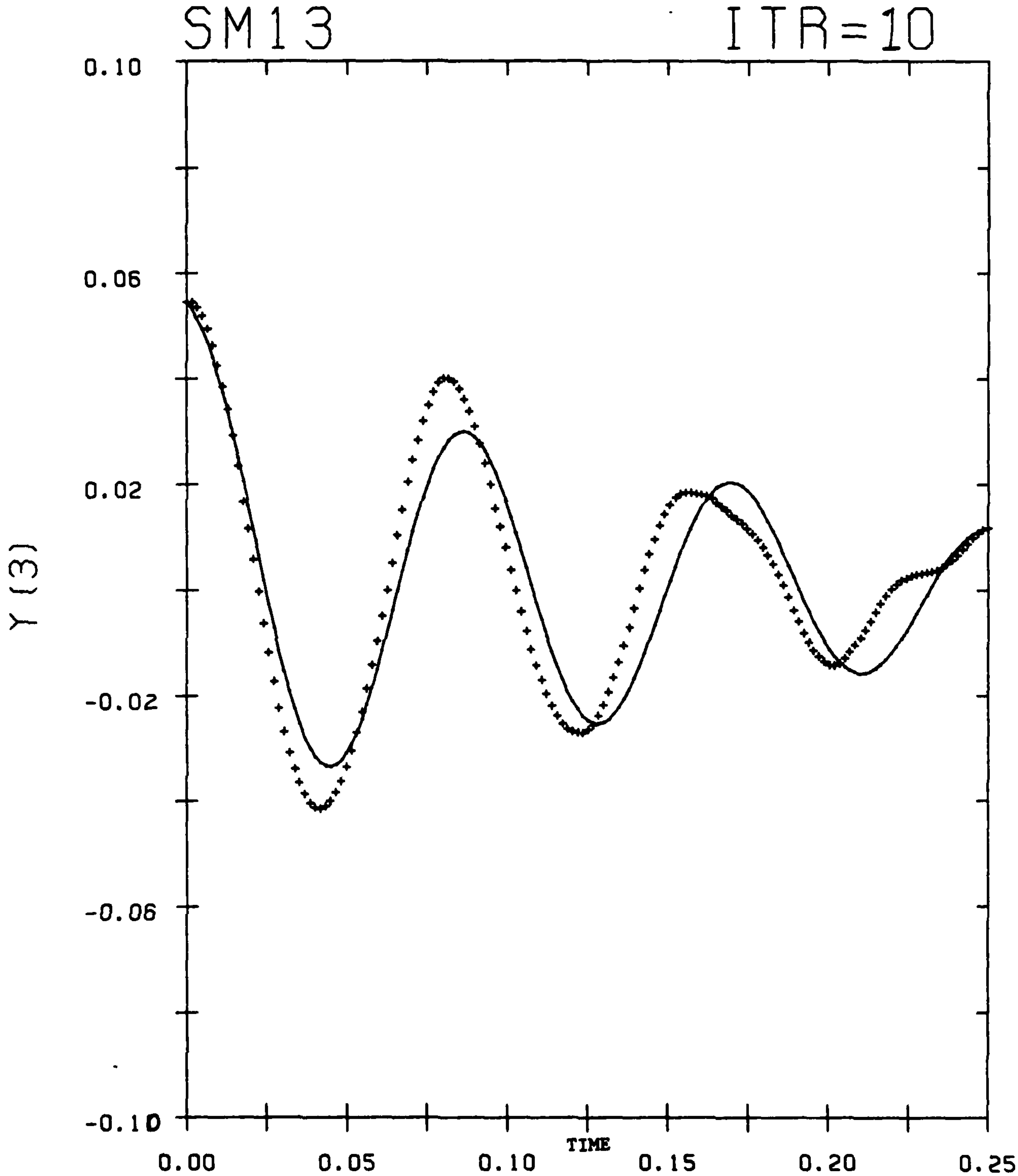

FIGURE 6.2.c. The converged data fits for the ordinary differential equation model; Example 6.1.

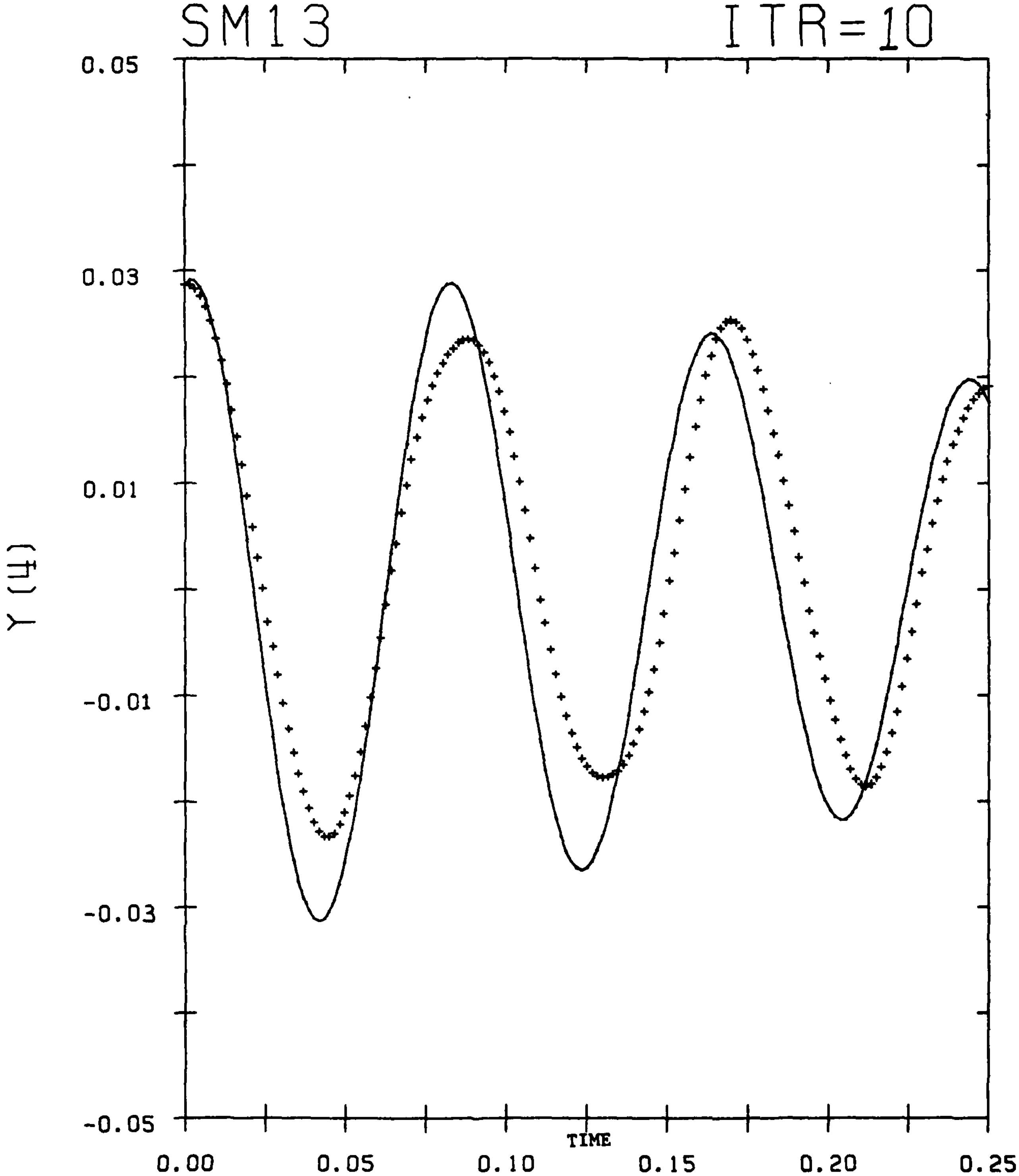

FIGURE 6.2.d. The converged data fits for the ordinary differential equation model; Example 6.1.

EXAMPLE 6.2

A second experiment was conducted using a simple dalay model for the unsteady aerodynamic behavior. The situation is very similar to the ordinary differential equation case described above, except that the matrix valued function $G(s;p)$ now has the form $G(s;p) = g_1 u_0(s) + g_2 u_{-r}(s)$ so that the Riemann-Stieltjes integral in equation (42) now becomes

$$\int_{-r}^{0} ds\, G(s;p) x_d(t+s) = g_1 x_d(t) + g_2 x_d(t-r) \ .$$

The system parameters identified include C_{ℓ_α}, C and g_1 as above, and the additional parameter g_2. The time-delay, r, was fixed at r = .05 sec. This selection was made based on a very preliminary scan of the data. In this case the requried initial data is the value $x_5(0)$ and the initial history of $x_5(\cdot)$ on the interval $[-r,0]$. Rather than allow the computer algorithm complete freedom of choice in determining the initial data several 'experiments' were performed with operator selected values. The values selected were $x_5(0) = 36.$, $x_5(-r/N) = 4.0$, $x_5(-2r/N) = .25$ and all other x_5 'knot' values are zero. In the results reported the SPLINE REDUCED procedure was used with N = 16.

TABLE 6.3. Parameter estimates for the hereditary model; Example 6.2.

iteration	C_{ℓ_α}	C	g_1	g_2
0	6.28	.500	-12.2	0
1	11.72	.739	-70.0	13.7
5	9.56	1.512	-270.5	-113.5
10	9.07	1.658	-309.8	-104.6
20	8.81	1.496	-274.8	-62.5
49	8.77	1.412	-255.1	-47.5
90	8.76	1.411	-254.8	-47.8

Results from the parameter identification algorithm for this case are shown in Table 6.3 and in Figure 6.3. The startup fits are the same as in the previous example and are not repeated. While the data fits for the hereditary model exhibit some improvement over the previous example the quality of the matches is still rather poor. As noted above it seems that one or more of the 'fixed' parameters has been incorrectly specified.

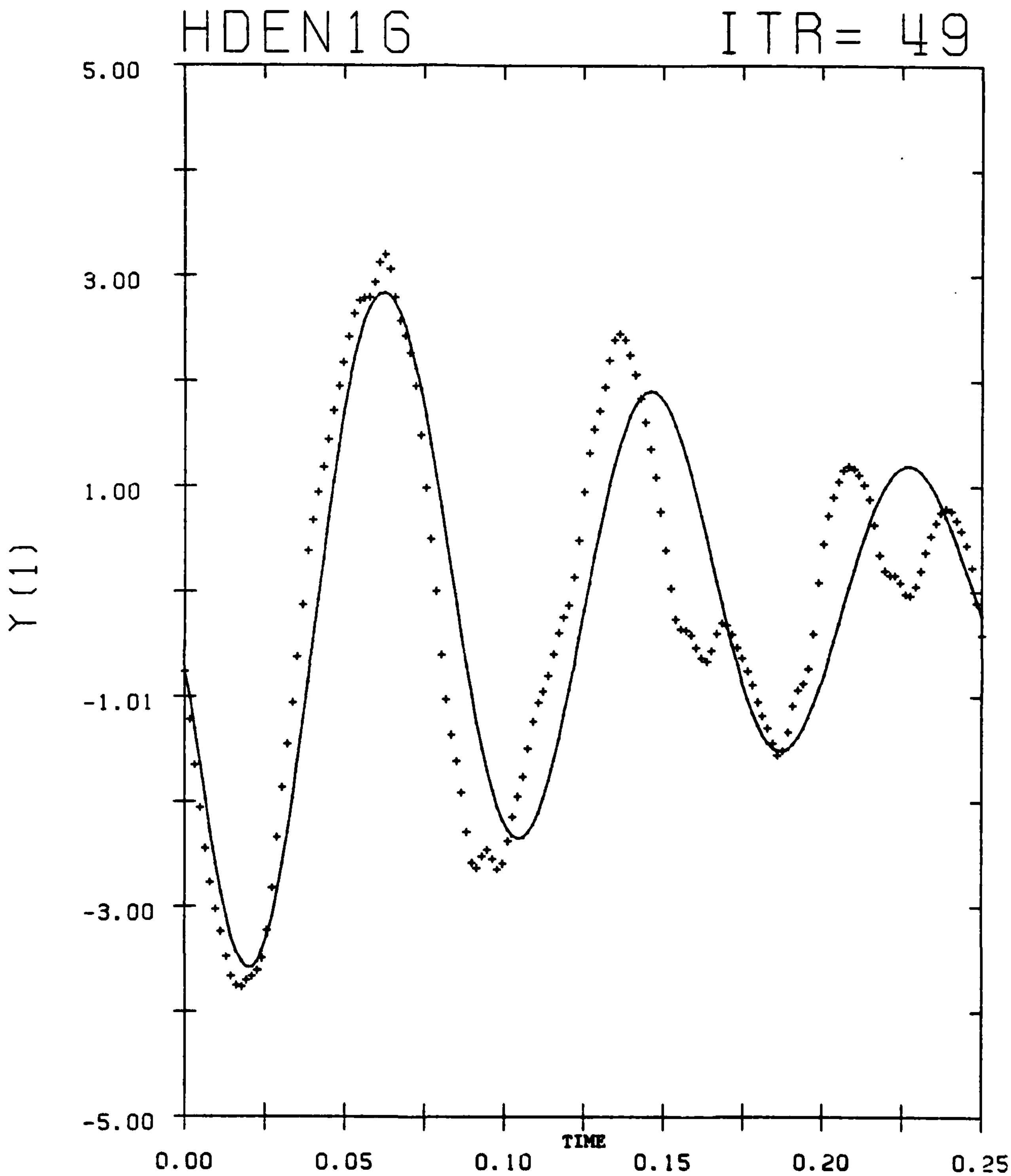

FIGURE 6.3. The converged data fits for the hereditary model; Example 6.2.

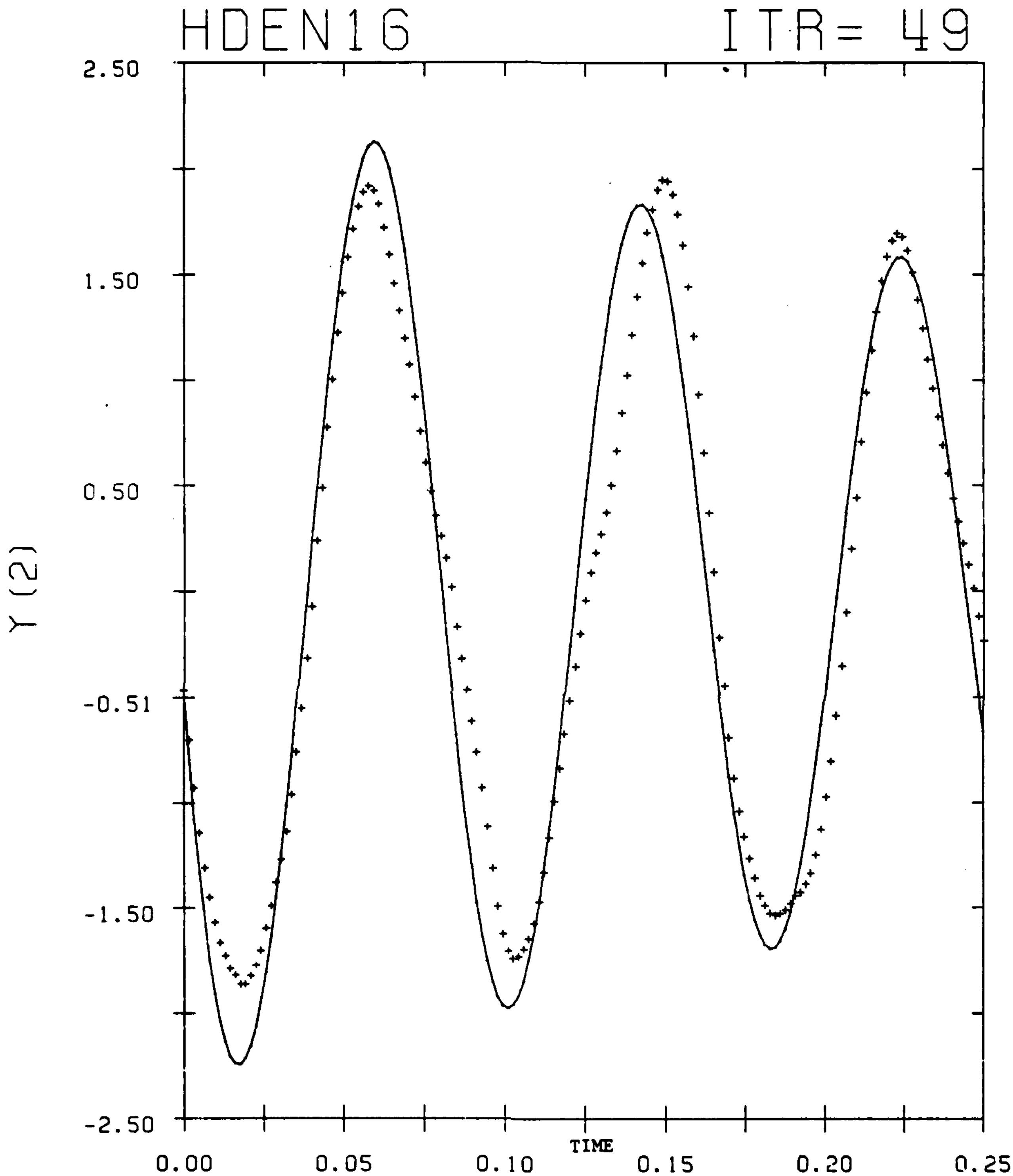

FIGURE 6.3.b. The converged data fits the hereditary model; Example 6.2.

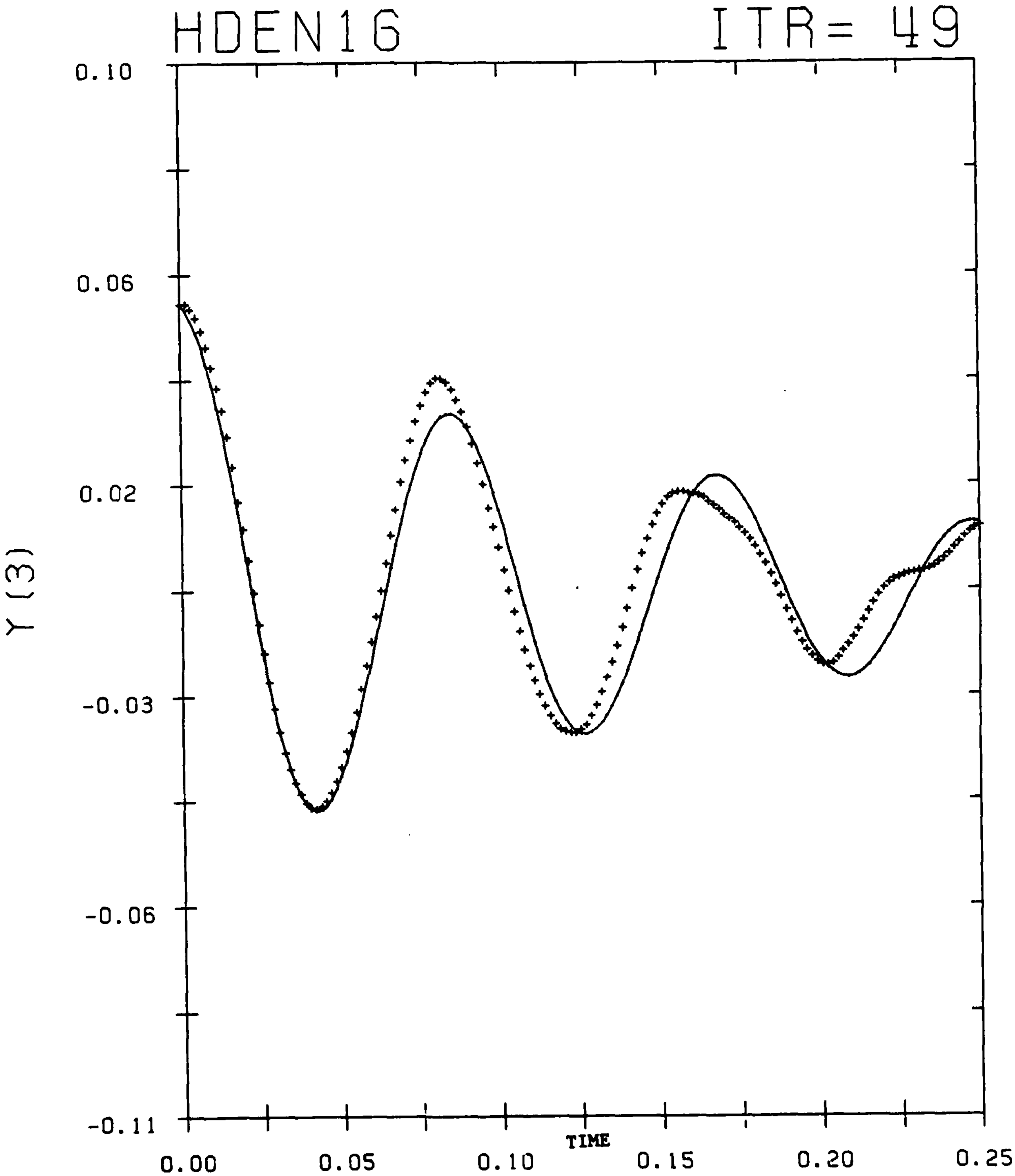

FIGURE 6.3.c. The converged data fits for the hereditary model; Example 6.2.

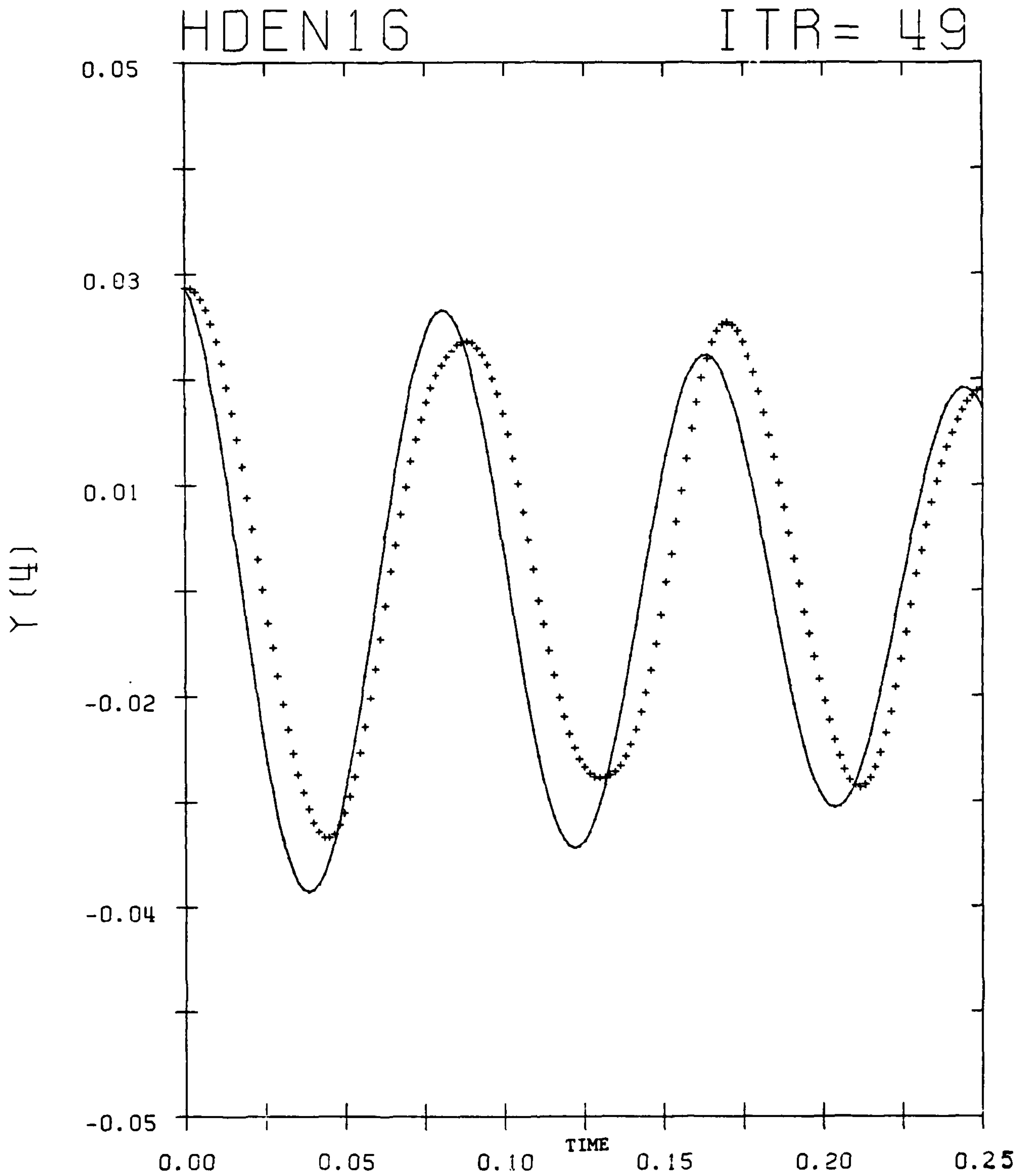

FIGURE 6.3.d. The converged data fits for the hereditary model; Example 6.2.

SECTION 7

CONCLUSIONS

The state space model (43) offers a generalization of earlier formulations of dynamic models for the system shown in Figure 1.1 [13, 16, 17]. The inclusion of a hereditary dynamic model for the 'aerodynamic' states is significant in that several recent investigations have shown that fundamental unsteady aerodynamic analysis produces precisely such hereditary models [1, 2, 9].

In order to deal efficiently with the problem of parameter estimation for the model (43) some earlier numerical algorithms for identification of hereditary systems [4, 5, 8] were reworked to incorporate separation of 'delayed' and 'non-delayed' states. This refinement results in significant savings in computer costs and increases the scope of problems for which the methods are easily implemented.

Some results from preliminary studies of identification using actual wind-tunnel data indicate some improvement when hereditary modelling is used. These results must be viewed with caution until more systematic studies are performed.

SECTION 8

REFERENCES

1. A. V. Balakrishnan, Active control of airfoils in unsteady aerodynamics, *Appl. Math. and Opt.*, *4*(1978), 171-195.

2. A. V. Balakrishman and J. W. Edwards, Calculation of the Transient motion of elastic airfoils forced by control surface motion and gusts, *NASA Technical Memorandum* 81351, August, 1980.

3. H. T. Banks and J. A. Burns, Hereditary control problems: numerical methods based on averaging approximations, *SIAM J. Control and Optimization*, *16*(1978), 169-208.

4. H. T. Banks, J. A. Burns and E. M. Cliff, Spline-based approximation methods for control and identification of hereditary systems, in *International Symposium on Systems Optimization and Analysis*, A. Bensoussan and J. L. Lions, eds., Lecture Notes in Control and Info. Sci., Vol. 14, Springer, Heidelberg, 1979, pp. 314-320.

5. H. T. Banks, J. A. Burns and E. M. Cliff, A comparison of numerical methods for identification and optimization problems involving control systems with delays, *Brown Univ. LCDS Tech. Rep.* 79-7, 1979, Providence, R.I.

6. R. G. Bartle, *The Elements of Real Analysis*, John Wiley & Sons, New York 1976.

7. R. Bellman and K. L. Cooke, *Differential Difference Equations*, Academic Press, New York, 1963.

8. J. A. Burns and E. M. Cliff, Parameter identification for hereditary systems: final technical report on grant AFOSR-77-3221A, September 1979. AFWAL TECHNICAL MEMORANDUM 80-10-FLGC, Wright-Patterson AFB, Ohio, 1980.

9. J. A. Burns and E. M. Cliff, Modeling unsteady aerodynamics via functional differential equations, preliminary report.

10. J. W. Edwards, Unsteady aerodynamic modeling and active aeroelastic control, *SUDAAR 504*, Stanford University Report, February, 1977.

11. J. W. Edwards, Unsteady aerodynamic modeling for arbitrary motions, *AIAA Journal*, *15*(1977), 593-595.

12. J. K. Hale, *Theory of Functional Differential Equations*, Springer-Verlag, New York, 1977.

13. R. T. Jones, Operational treatment of the nonuniform lift theory to airplane dynamics, *NACA TN 667*, 1938.

14. J. P. Reding and L. E. Ericsson, Aeroelastic stability of the 747/Orbiter, *J. Aircraft*, 14(1977), 988-993.

15. J. P. Reding and L. E. Ericsson, Effects of flow separation on shuttle longitudinal dynamics and aeroelastic stability, *J. Spacecraft*, 14(1977), 711-718.

16. W. P. Rodden and B. Stahl, A strip method for prediction of damping in subsonic wind tunnel flight test, *J. of Aircraft*, *6*(1969), 9-17.

17. D. L. York, Analysis of flutter and flutter suppression via an energy method, MS Thesis, Aerospace and Ocean Engineering Department, Virginia Polytechnic Institute and State University, Blacksburg, VA, June, 1980.